Macroscopic Loop Amplitudes in the Multi-Cut Matrix Models

Hirotaka Irie

(National Taiwan University, Taiwan)

@ IST Thematic Period,
"Matrix Models and Geometry"

Ref)

HI, "Fractional supersymmetric Liouville theory and the multi-cut matrix models", **Nucl. Phys. B819 [PM] (2009) 351**

CT-Chan, HI,SY-Shih and CH-Yeh, "Macroscopic loop amplitudes in the multi-cut two-matrix models," **Nucl. Phys. B (10.1016/j.nuclphysb.2009.10.017),** in press

CT-Chan, HI,SY-Shih and CH-Yeh, "Macroscopic loop amplitudes in the Z_k symmetry breaking critical points of the multi-cut two-matrix models," in preparation.

Collaborators of [CISY]:

[CISY1] “Macroscopic loop amplitudes in the multi-cut two-matrix models”, **Nucl. Phys. B (10.1016/j.nuclphysb.2009.10.017)**

[CISY2] “Macroscopic loop amplitudes in the Z_k symmetry breaking critical points of the multi-cut two-matrix models”, in preparation

Chuan-Tsung Chan
(Tunghai University, Taiwan)

Sheng-Yu Darren Shih
(NTU, Taiwan → UC, Berkeley, US [Sept. 1 ~])

Chi-Hsien Yeh
(NTU, Taiwan)

What is the multi-cut matrix model ?

TWO-matrix model $\mathcal{Z} = \int dXdY\, e^{-Ntr[V_1(X)+V_2(Y)-XY]}$

(If V(x) is gaussian => ONE-matrix $\mathcal{Z} = \int dM\, e^{-NtrV(M)}$)

Matrix ⇔ String   Feynman Graph = Spin models on Dynamical Lattice

TWO-matrix model ➔ **Ising model** on Dynamical Lattice

Continuum theory at the simplest critical point $V_1(x)\left(= V_2(x)\right) = -3x - \frac{3}{2}x^2 + \frac{x^3}{3}$

Critical Ising models coupled to 2D Worldsheet Gravity

(3,4) minimal CFT ⊗ Liouville field theory ⊗ conformal ghost

Continuum theories at **multi-critical** points

(p,q) minimal CFT ⊗ Liouville field theory ⊗ conformal ghost

(p,q) minimal string theory

The two-matrix model includes more !

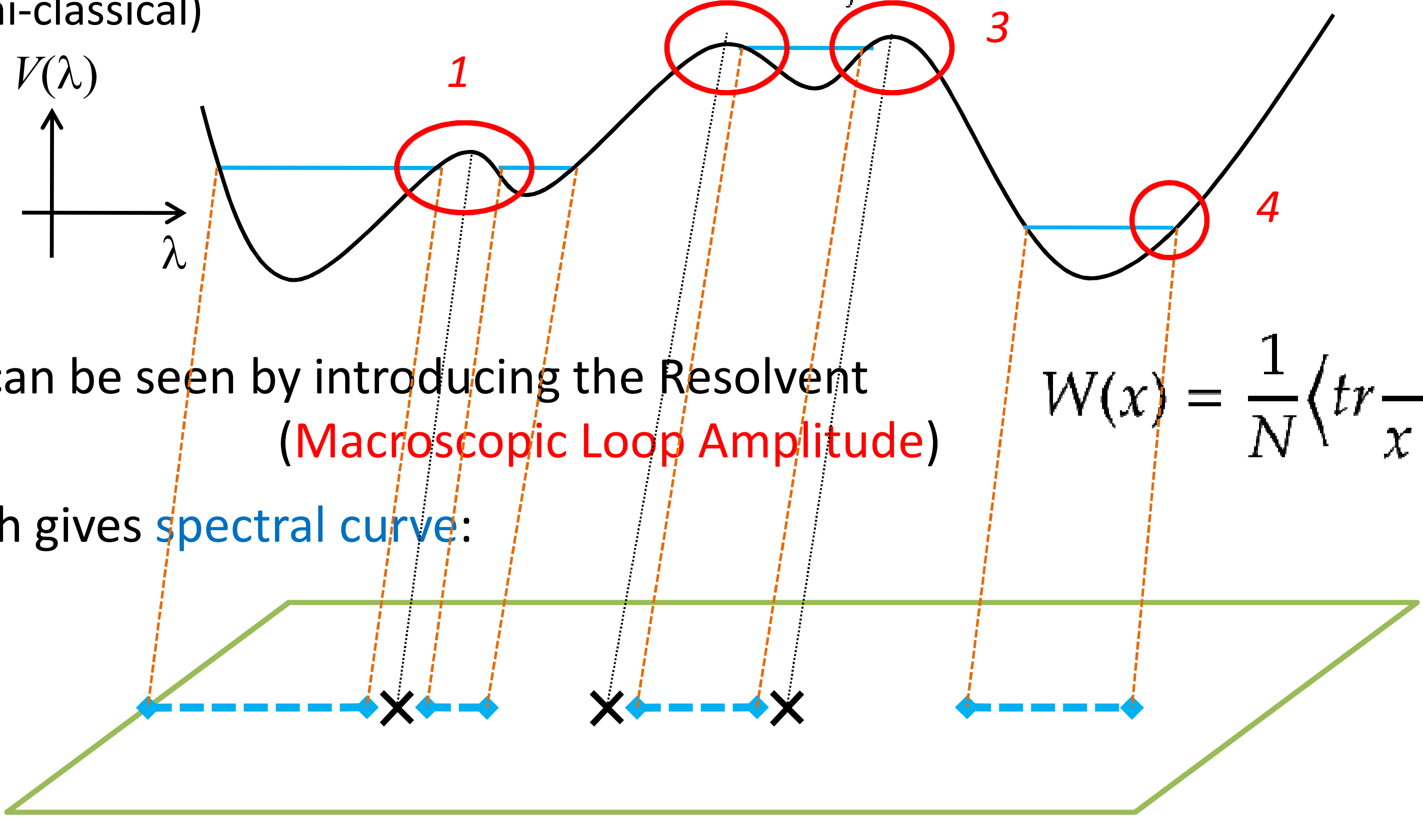
CUTs

Let's see more details

Diagonalization: $U^{\dagger}MU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$

$$Z = \int d^N\lambda \prod_{i>j} (\lambda_i - \lambda_j)^2 e^{-N\sum_i V(\lambda_i)}$$

In Large N limit
(= semi-classical)



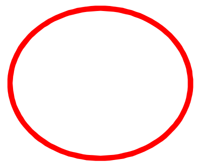
This can be seen by introducing the Resolvent
(Macroscopic Loop Amplitude)

$$W(x) = \frac{1}{N}\left\langle tr \frac{1}{x-M} \right\rangle$$

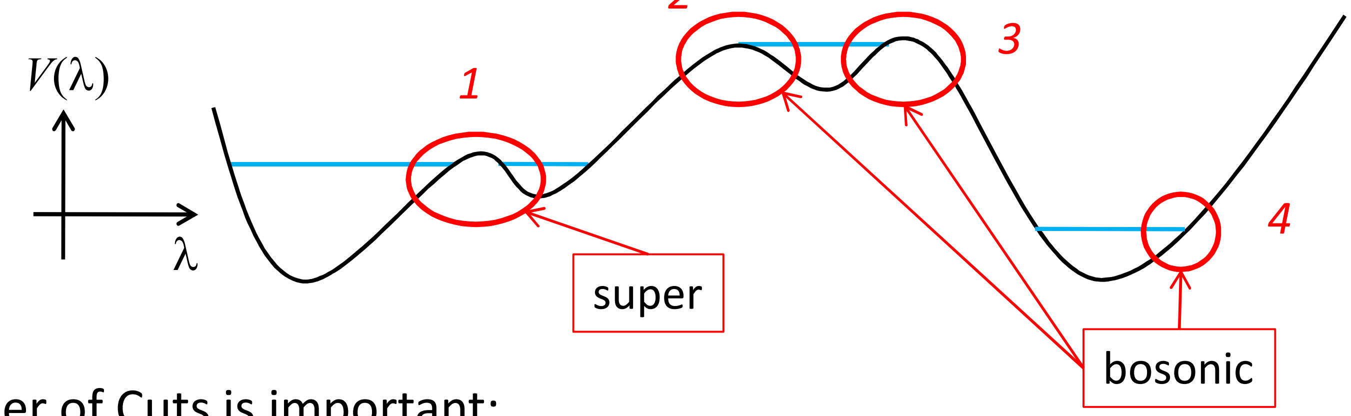
which gives spectral curve:

Continuum limit = Blow up *some points of x*

The nontrivial things occur only around the turning points



Correspondence with string theory



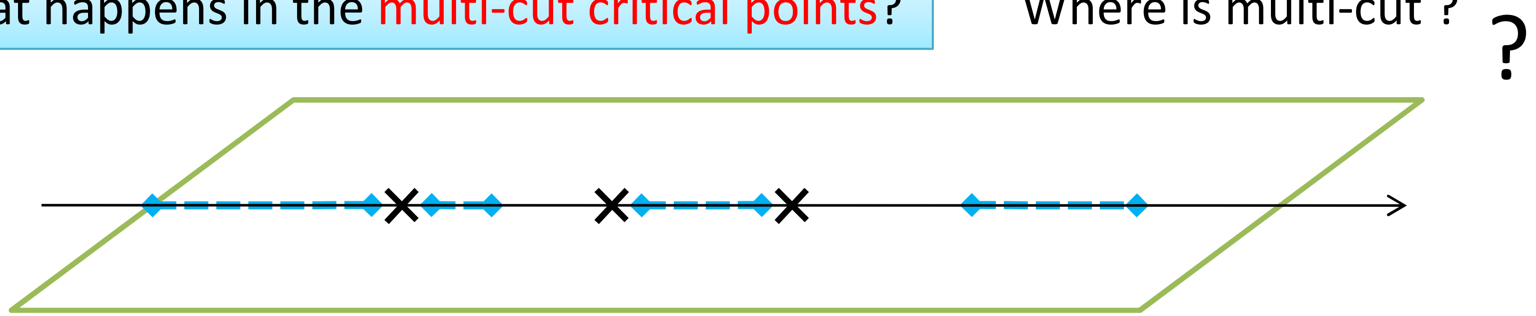
<u>The number of Cuts is important:</u>

- 1-cut critical points (2, 3 and 4) give *(p,q) minimal (bosonic) string theory*
- 2-cut critical point (1) gives *(p,q) minimal superstring theory*

[Takayanagi-Toumbas '03], [Douglas et.al. '03], [Klebanov et.al '03]

What happens in the multi-cut critical points?

Where is multi-cut ?



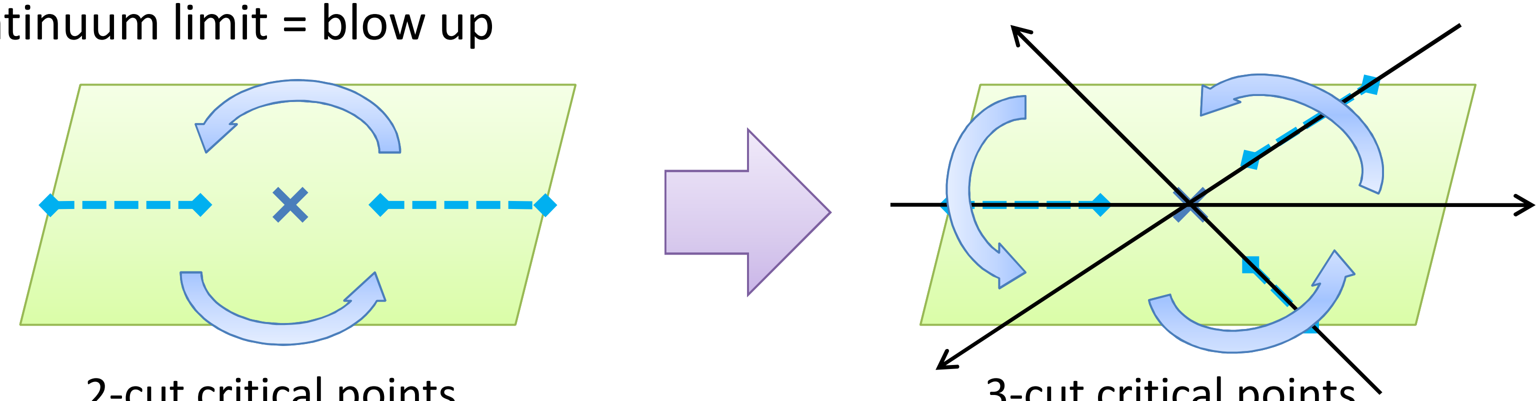
Continuum limit = blow up

Maximally TWO cuts *<u>around the critical points</u>*

How to define the *Multi-Cut* *Critical Points*:

[Crinkovik-Moore '91]

Continuum limit = blow up



The matrix Integration is defined by the normal matrix:

$$\mathcal{Z} = \int dXdY\, e^{-N tr[V_1(X)+V_2(Y)-XY]}$$

$$\left(X, Y = U\,\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)\, U^\dagger \quad U \in U(N), \quad \lambda_i \in \mathbb{R}\, e^{2\pi \frac{n}{k}} (n \in \mathbb{Z}) \right)$$

We consider that this system naturally has Z_k transformation:

$$x \to \omega x \quad (\omega = e^{\frac{2\pi i}{k}})$$

Why is it good?

This system is controlled by k-comp. KP hierarchy

[Fukuma-HI '06]

A brief summary for multi-comp. KP hierarchy

[Fukuma-HI '06]

Lax op. comes from the recursive relations of the orthonormal polynomial system:

$$\delta_{n,m} = \int dX\, dY\, e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}\, \alpha_n(x)\, \beta_m(y)$$

Z_k –symmetric potentials: $w(\omega X, \omega^{-1} Y) = w(X,Y) = V_1(X) + V_2(Y) - XY$

ASSUME: the critical point is at the origin

k distinct orthonormal polynomials at the origin:

$$\alpha_n(\omega x) = \omega^n \alpha_n(x)$$

somehow →

k distinct smooth continuum functions at the origin:

$$f^{(r)}(x;l) \quad (r = 0, 1, \cdots, k-1)$$

Smooth function in x and l

$$n = kl + r$$

Recursive relations:

$$x\,\alpha_n(x) = \hat{A}(n; e^{\partial_n})\,\alpha_n(x) = A_{-1}(n)\,\alpha_{n+1}(x) + A_0(n)\,\alpha_n(x) + \cdots$$

$$N^{-1}\frac{\partial}{\partial x}\alpha_n(x) = \hat{B}(n; e^{\partial_n})\,\alpha_n(x) = B_1(n)\,\alpha_{n-1}(x) + B_2(n)\,\alpha_{n-2}(x) + \cdots$$

$$\psi(x;l) = \begin{pmatrix} f^{(0)}(x;l) \\ f^{(1)}(x;l) \\ \vdots \\ f^{(k-1)}(x;l) \end{pmatrix}$$

k x k matrix op.

k-comp. KP lax op.

⇒

$$x\,\psi(x;l) = \mathcal{A}(l, e^{\partial_l}) \cdot \psi(x;l)$$

$$N^{-1}\frac{\partial}{\partial x}\psi(x;l) = \mathcal{B}(l, e^{\partial_l}) \cdot \psi(x;l)$$

Cont. ⇒

$$\mathcal{A}(l; e^{\partial_l}) = C^{\mathcal{A}}\,\partial_l^{\hat{p}} + O(\partial_l^{\hat{p}+1})$$

$$\mathcal{B}(l; e^{\partial_l}) = C^{\mathcal{B}}\,\partial_l^{\hat{q}} + O(\partial_l^{\hat{q}+1})$$

Scaling ansatz and critical potentials and hermiticity [CISY ‘09]

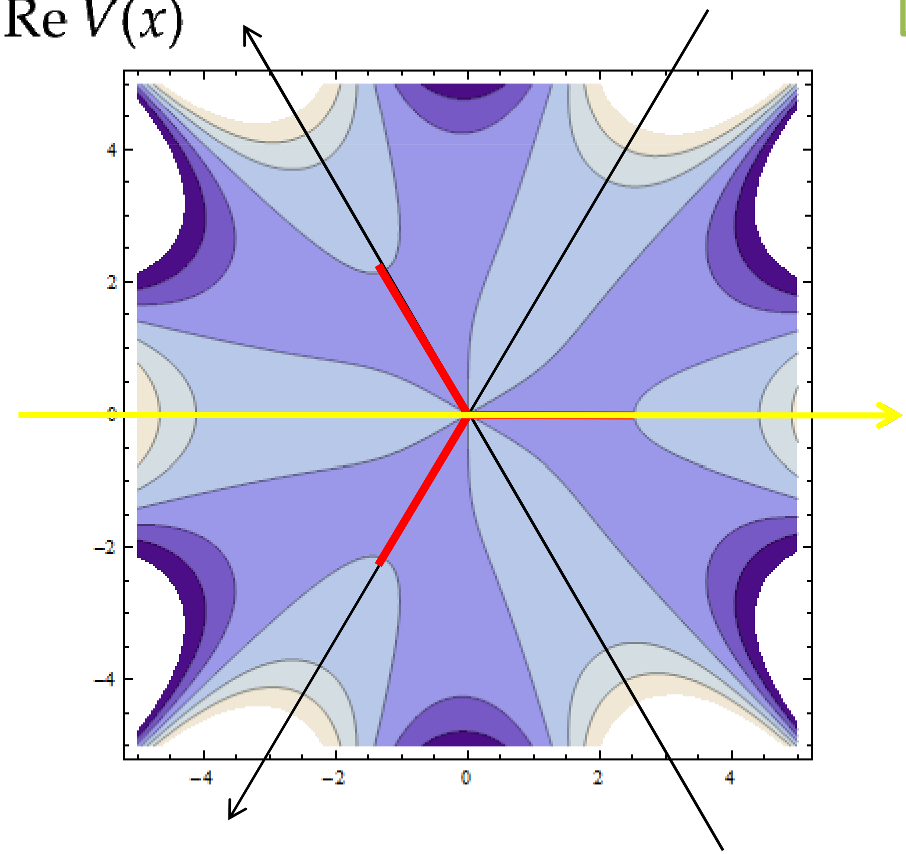
The relation between α and f: [Crinkovik-Moore ‘91](ONE), [CISY ‘09](TWO)

$$\alpha_{kl+r}(x) = (-1)^l \omega^{-(r/2)} f^{(r)}(x,l)$$

From this, we can obtain critical potentials and so on...

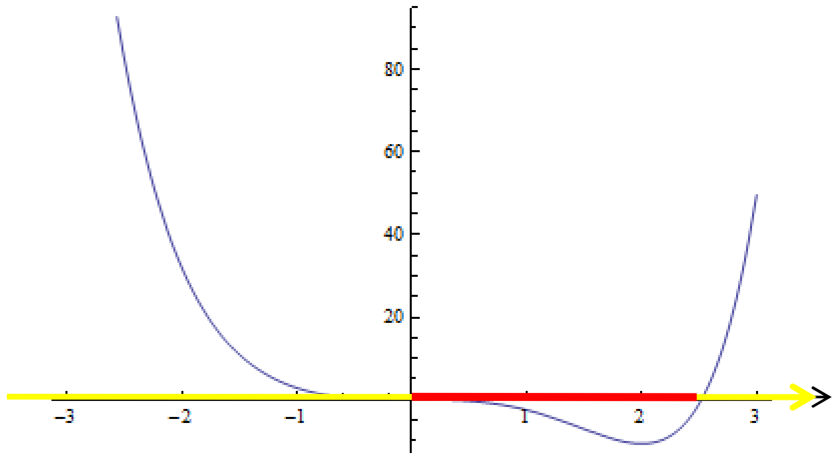
e.g.) (2,2) 3-cut critical point [CISY ‘09]

$\mathrm{Re}\, V(x)$



Critical potential

$$V_1(x) = V_2(x) = -\frac{8}{3}x^3 + \frac{x^6}{6}$$



Blue = lower / yellow = higher

Scaling ansatz and critical potentials and hermiticity

[CISY '09]

The relation between α and f: [Crinkovik-Moore '91](ONE), [CISY '09](TWO)

$$\alpha_{kl+r}(x) = (-1)^l \omega^{-(r/2)} f^{(r)}(x,l)$$

NOTE [CISY '09]

The operators A and B are given by

$$\mathcal{A}(l;e^{\partial_l}) = C_0^{\mathcal{A}}\,\partial_l^{\hat{p}} + C_1^{\mathcal{A}}(l)\,\partial_l^{\hat{p}-1} + \cdots + C_{\hat{p}}^{\mathcal{A}}(l)$$

$$\mathcal{B}(l;e^{\partial_l}) = C_0^{\mathcal{B}}\,\partial_l^{\hat{q}} + C_1^{\mathcal{B}}(l)\,\partial_l^{\hat{q}-1} + \cdots + C_{\hat{q}}^{\mathcal{B}}(l)$$

1. The coefficient matrices are all REAL
2. The Z_k symmetric critical points are given by

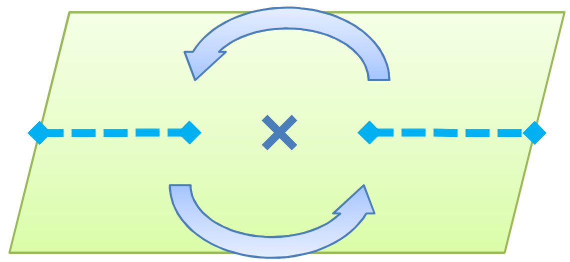
$$C_0^{\mathcal{A}} = \Gamma, \quad C_0^{\mathcal{B}} = \Gamma^{-1} \qquad \Gamma = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix}$$

(Γ : the shift matrix)

3. We can also break Z_k symmetry (difference from Z_k symmetric cases)

$$C_0^{\mathcal{A}} = \Gamma, \quad C_0^{\mathcal{B}} = \Gamma, \text{ or } (\Gamma + \Gamma^{-1}), \text{ or } \cdots$$

What should correspond to the multi-cut matrix models? [HI '09]



2-cut critical points

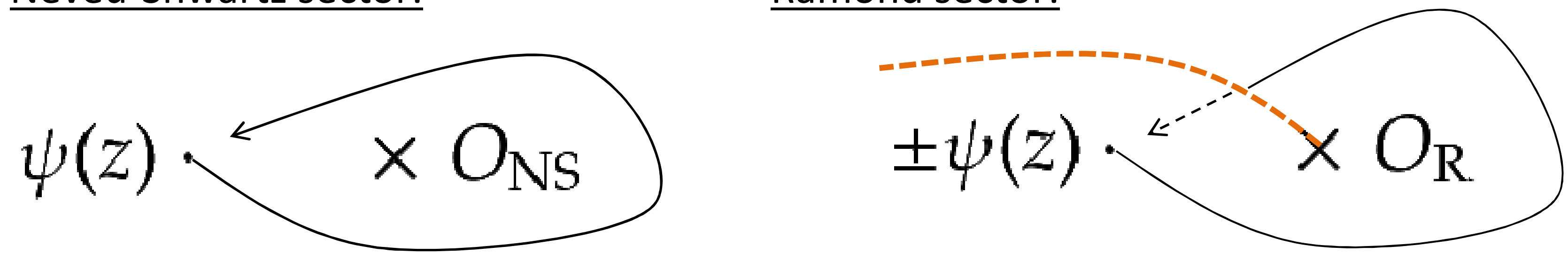
Reconsider the TWO-cut cases:

Z_2 transformation

= Z_2 R-R charge conjugation

[Takayanagi-Toumbas '03], [Douglas et.al. '03], [Klebanov et.al '03]

Because of WS Fermion



Z_2 spin structure => Selection rule => Charge

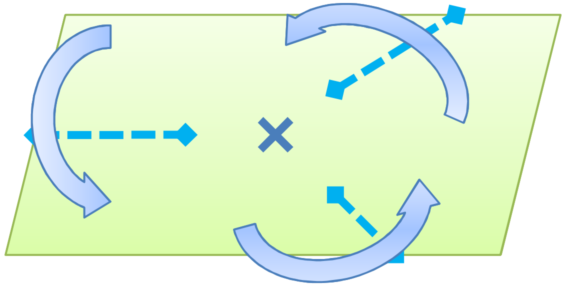
ONE-cut: $X(z)$ ⇒ TWO-cut: $X(z) + \psi(z)$

Gauge it away by superconformal symmetry

Superstrings !

What should correspond to the multi-cut matrix models? [HI '09]

How about **MULTI (=k) -cut** cases: [HI '09]



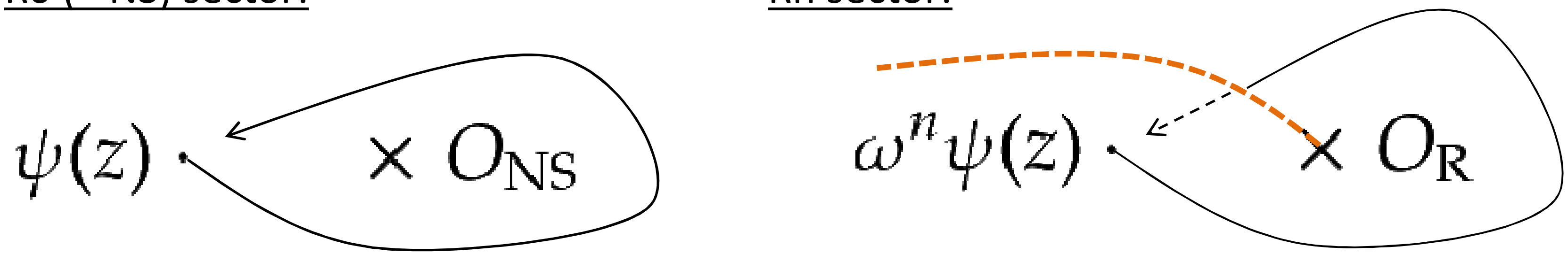
3-cut critical points

Z_k transformation
=?= Z_k "R-R charge" conjugation

Because of "Generalized Fermion"

RO (= NS) sector:

Rn sector:



Z_k spin structure => Selection rule => Charge

ONE-cut: $X(z)$ ⇒ multi-cut: $X(z) + \psi(z)$

Gauge it away by Generalized superconformal symmetry

Which ψ ? [HI '09]

Zamolodchikov-Fateev parafermion

[Zamolodchikov-Fateev '85]

1. Basic parafermion fields: $\psi(z)$

2. Central charge: $c_{Z_k} = \frac{2(k-1)}{k+2}$

3. The OPE algebra: $\left[\psi(z)\right]^l = \left[\psi_l(z)\right]$

k parafermion fields: $\psi_1 (= \psi),\ \psi_2, \cdots, \psi_k (= 1)$

Z_k spin-fields: $\sigma_\lambda(z)\ (\lambda = 0, 1, \cdots, k-1)$

4. The symmetry and its minimal models of $X(z) + \psi(z)$:

k-Fractional superconformal sym.

GKO construction

$$\frac{SU(2)_k \otimes SU(2)_l}{SU(2)_{k+l}}$$

Minimal k-fractional super-CFT:

(k=1: bosonic, k=2: super)

Why is it good?

It is because *the spectrums* of
the (p,q) minimal fractional super-CFT
and *the multi-cut matrix models*
are *the same*. [HI '09]

Correspondence [HI '09]

(p,q) Minimal k-fractional super-CFT $(p,q)=(\hat{k}\hat{p},\hat{k}\hat{q}) \quad (\hat{p},\hat{q}: \text{coprime})$

$$k=\hat{k}\times d_{\hat{q}-\hat{p}}, \quad d_{\hat{q}-\hat{p}}=\text{m.c.d}\{k,\hat{q}-\hat{p}\}$$

$$\mathcal{A}(l,e^{\partial_l})=\Gamma\,\partial_l^{\hat{p}}+\cdots, \quad \mathcal{B}(l,e^{\partial_l})=\Gamma\,\partial_l^{\hat{q}}+\cdots$$

Z_k symmetric case:

$$\mathcal{A}(l,e^{\partial_l})=\Gamma\,\partial_l^{\hat{p}}+\cdots, \quad \mathcal{B}(l,e^{\partial_l})=\Gamma^{-1}\,\partial_l^{\hat{q}}+\cdots$$

This means that Z_k symmetry is broken (upto Z_2) !

Minimal k-fractional super-CFT requires Screening charge:

$$Q^{(+)}=\oint dz\,\psi(z)\,e^{i(2b/k)X(z)}, \quad \text{: R2 sector}$$

$$Q^{(-)}=\oint dz\,\psi_{k-1}(z)\,e^{-i(2/bk)X(z)} \quad \text{: R(k-2) sector}$$

Z_k sym is broken (upto Z_2) !

Summary of Evidences and Issues [HI '09]

- ✓ Labeling of critical points (p,q) and Operator contents
  (ASSUME: OP of minimal CFT and minimal strings are 1 to 1)
- ✓ Residual symmetry
  → On both sectors, the Z_k symmetry broken or to Z_2.
- ✓ String susceptibility (of cosmological constant)

Q1 ☐ Fractional supersymmetric Ghost system
→ bc ghost + "ghost chiral parafermion" ?

Q2 ☐ Covariant quantization / BRST Cohomology
→ Complete proof of operator correspondence

Q3 ☐ Correlators
(D-brane amplitudes / macroscopic loop amplitudes)
→ Ghost might be factorized (we can compare Matrix/CFT !)

Let's consider macroscopic loop amplitudes !!

Macroscopic Loop Amplitudes

Role of Macroscopic loop amplitudes

$$W(x) = \frac{1}{N}\left\langle tr \frac{1}{x-M} \right\rangle$$

1. Eigenvalue density $\rho(\lambda)$
2. Generating function of On-shell Vertex Op: trM^n
3. FZZT-brane disk amplitudes (Comparison to String Theory)
4. D-Instanton amplitudes also come from this amplitude

1-cut general (p,q) critical point: [Kostov '90]

$$x = \sqrt{\mu}\cosh p\tau = \sqrt{\mu}T_p(z), \quad W = \mu^{q/2p}\cosh q\tau = \mu^{q/2p}T_q(z)$$

$\cosh p\tau = T_p(\cosh(\tau)), \quad z = \cosh(\tau)$:the Chebyshev Polynomial of the 1st kind

2-cut general (p,q) critical point: [Seiberg-Shih '03] (from Liouville theory)

$$x = \sqrt{\mu}\sinh p\tau = \sqrt{\mu}U_p(z)\sqrt{z^2-1}, \quad W = \mu^{q/2p}\sinh q\tau = \mu^{q/2p}U_q(z)\sqrt{z^2-1}$$

$\sinh p\tau = U_{p-1}(\cosh(\tau))\sinh(\tau)$:the Chebyshev Polynomial of the 2nd kind

The Daul-Kazakov-Kostov prescription (1-cut) [DKK '93]

$$W(x) = \frac{1}{N}\left\langle tr \frac{1}{x-M} \right\rangle \quad \to \quad \Psi_n(x) = \left\langle \det(x-M) \right\rangle_{n\times n}$$

$\Psi_n(x)$: Orthogonal polynomial [Gross-Migdal '90]

$$x\Psi_n(x) = \hat{A}(n; e^{\partial_n})\Psi_n(x), \quad N^{-1}\frac{d}{dx}\Psi_n(x) = \hat{B}(n; e^{\partial_n})\Psi_n(x)$$

$$x \sim \mathbf{P}(t;\partial) = a^{-p/2}\hat{A}(n; e^{\partial_n}) = C_0^A\,\partial^p + C_1^A(t)\,\partial^{p-1} + \cdots + C_p^A(t)$$

$$W \sim \mathbf{Q}(t;\partial) = a^{-q/2}\hat{B}(n; e^{\partial_n}) = C_0^B\,\partial^q + C_1^B(t)\,\partial^{q-1} + \cdots + C_q^B(t)$$

$$[\mathbf{P}, \mathbf{Q}] = g_s$$

Solve this system at the week coupling limit

More precisely [DKK '93]

$$\partial_n \to a^{1/2}\partial \equiv a^{1/2} g_s\,\partial_t, \quad N^{-1} = a^{(p+q)/2} g_s, \quad n/N = e^{-a^{(p+q-1)/2}t}$$

The Daul-Kazakov-Kostov prescription (1-cut) [DKK '93]

$$x \sim \mathbf{P}(t;\partial) = a^{-p/2}\hat{A}(n;e^{\partial_n}) = C_0^A\,\partial^p + C_1^A(t)\,\partial^{p-1} + \cdots + C_p^A(t)$$

$$W \sim \mathbf{Q}(t;\partial) = a^{-q/2}\hat{B}(n;e^{\partial_n}) = C_0^B\,\partial^q + C_1^B(t)\,\partial^{q-1} + \cdots + C_q^B(t)$$

$$[\mathbf{P},\mathbf{Q}] = g_s$$

Solve this system at the week coupling limit

Scaling

Dimensionless valuable: $z \equiv \bar{g}_s\, t\, \partial_t$

$$\mathbf{P}(t;\partial) = t^{\frac{p}{p+q-1}}\,\tilde{T}_p(z), \qquad \mathbf{Q}(t;\partial) = t^{\frac{q}{p+q-1}}\,\tilde{T}_q(z),$$

$$[\mathbf{P},\mathbf{Q}] = g_s \quad \Leftrightarrow \quad q\tilde{T}_p'(z)\tilde{T}_q(z) - p\tilde{T}_q'(z)\tilde{T}_p(z) = const$$

Eynard-Zinn-Justin-Daul-Kazakov-Kostov eq.

$$(\tilde{T}_p(\cosh\tau) \sim \cosh p\tau,\ \ z \sim \cosh\tau)$$

Addition formula of trigonometric functions (q=p+1: Unitary series)

$$[pq/\sinh(q-p)\tau] \times (\sinh p\tau \cosh q\tau - \sinh q\tau \cosh p\tau) = const$$

The k-Cut extension [CISY '09]

$$x \simeq \mathbf{P}(t;\partial) = a^{-\hat{p}/2}\mathcal{A}(l;e^{\partial_l}) = C_0^{\mathcal{A}}\,\partial^{\hat{p}} + C_1^{\mathcal{A}}(t)\,\partial^{\hat{p}-1} + \cdots + C_{\hat{p}}^{\mathcal{A}}(t)$$

$$W \simeq \mathbf{Q}(t;\partial) = a^{-\hat{q}/2}\mathcal{B}(l;e^{\partial_l}) = C_0^{\mathcal{B}}\,\partial^{\hat{q}} + C_1^{\mathcal{B}}(t)\,\partial^{\hat{q}-1} + \cdots + C_{\hat{q}}^{\mathcal{B}}(t)$$

$$[\mathbf{P},\mathbf{Q}] = g_s I_k$$

Solve this system in the week coupling limit

The coefficients $C_i^{\mathcal{A}}$ and $C_i^{\mathcal{B}}$ are *k x k matrices*

1. The 0-th order: $[\mathbf{P},\mathbf{Q}] = 0$: Simultaneous diagonalization

$$\mathbf{P}(t;\partial) \to t^{\frac{\hat{p}}{\hat{p}+\hat{q}-1}}\, P_p^{(j)}(z), \quad \mathbf{Q}(t;\partial) \to t^{\frac{\hat{q}}{\hat{p}+\hat{q}-1}}\, Q_q^{(j)}(z),$$

2. The next order: (eigenvalues: j=1, 2, … , k)

$$qP_p^{(j)\prime}(z)Q_q^{(j)}(z) - pQ_q^{(j)\prime}(z)P_p^{(j)}(z) = const$$

EZJ-DKK eq.

they are no longer polynomials in general

Let's consider the Z_k symmetric case: $C_0^{\mathcal{A}} = \Gamma, \quad C_0^{\mathcal{B}} = \Gamma^{-1}$

The Z_k symmetric cases can be restricted to

$$\mathcal{A}(l;e^{\partial_l}) \sim \mathbf{P}(t;\partial) = \begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & \ddots & \\ & & & 0 & * \\ * & & & & 0 \end{pmatrix}$$

$$\mathcal{B}(l;e^{\partial_l}) \sim \mathbf{Q}(t;\partial) = \begin{pmatrix} 0 & & & & * \\ * & 0 & & & \\ & \ddots & \ddots & & \\ & & * & 0 & \\ & & & * & 0 \end{pmatrix}$$

Our Ansatz: [CISY '09]

$$\mathbf{P}(\partial_n) = \sqrt{\mu}\,\pi_{p-1}(z) \begin{pmatrix} 0 & \overbrace{z-b & & }^{l} & & & \\ & 0 & \ddots & & & & \\ & & \ddots & z-b & & & \\ & & & 0 & z-a & & \\ & & & & \ddots & \ddots & \\ & & & & & 0 & z-a \\ z-a & & & & & & 0 \end{pmatrix}$$

$$\mathbf{Q}(\partial_n) = \mu^{\frac{q}{2p-1}}\,\xi_{q-1}(z) \begin{pmatrix} 0 & & & & & & z-b \\ z-a & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & \underbrace{& & z-a}_{l} & 0 & & & \\ & & & z-b & \ddots & & \\ & & & & \ddots & 0 & \\ & & & & & z-b & 0 \end{pmatrix}$$

The EZJ-DKK eq. gives several polynomials…

(k,l)=(3,1) cases [CISY '09]

$$\pi_{\hat{p}-1}(z) = \rho_{\hat{p}}(z)$$

$$\rho_2(z) = (-3)^2\left(z + \frac{c}{3}\right),$$

$$\rho_3(z) = (-3)^3\left(z^2 + \frac{c}{3}z - \frac{8c^2}{27}\right),$$

$$\rho_4(z) = (-3)^4\left(z^3 + \frac{c}{3}z^2 - \frac{5c^2}{9}z - \frac{7c^3}{81}\right),$$

$$\rho_5(z) = (-3)^5\left(z^4 + \frac{c}{3}z^3 - \frac{17c^2}{21}z^2 - \frac{97c^3}{567}z + \frac{128c^4}{1701}\right),$$

$\ldots$

We identify $\pi_p(z)$ *and* $\xi_q(z)$ are the Jacobi Polynomial

Our Solution: [CISY '09]

$$\mathbf{P} \simeq \sqrt{\mu}\, P_{p-1}^{\left(\frac{2l-k}{k},-\frac{2l-k}{k}\right)}(z/\theta)\, \sqrt[k]{(z-\theta)^{l}(z+\theta)^{k-l}} \times \Omega$$

$$\mathbf{Q} \simeq \mu^{\frac{q}{2p-1}}\, P_{q-1}^{\left(-\frac{2l-k}{k},\frac{2l-k}{k}\right)}(z/\theta)\, \sqrt[k]{(z-\theta)^{k-l}(z+\theta)^{l}} \times \Omega^{-1},$$

Our ansatz is applicable to general k (=3,4,…) cut cases
And every case is written with the Jacobi Polynomial

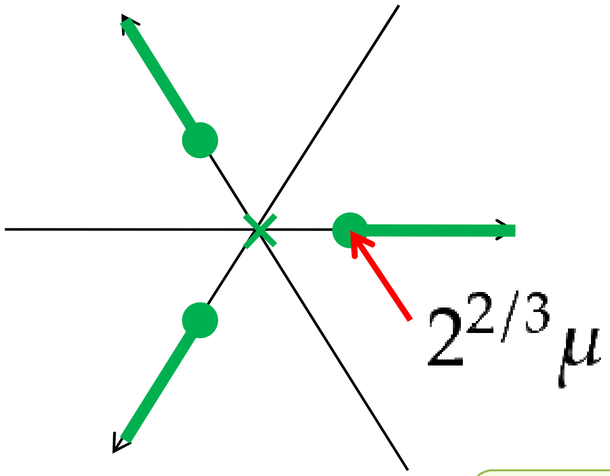
$$\int_{-1}^{1} dz\, P_n^{(\alpha,\beta)}(z)\, P_m^{(\alpha,\beta)}(z)\,(1-z)^{\alpha}(1+z)^{\beta} = const.\,\delta_{n,m}.$$

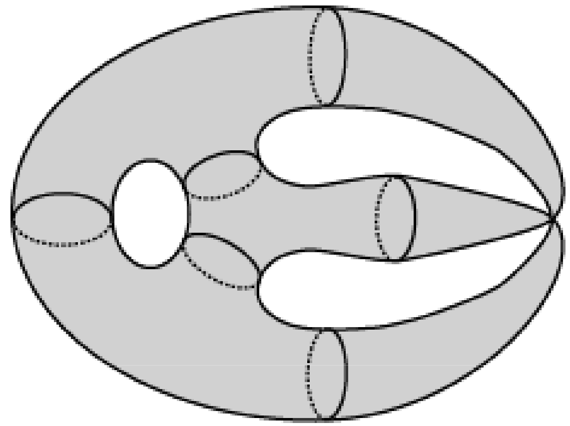
$P_n^{(-1/2,-1/2)}(z) = const.\, T_n(z),$: 1st Chebyshev

$P_n^{(1/2,1/2)}(z) = const.\, U_n(z),$: 2nd Chebyshev

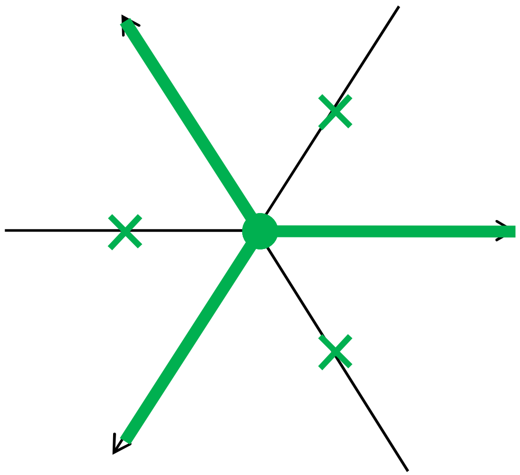
Geometry [CISY '09]

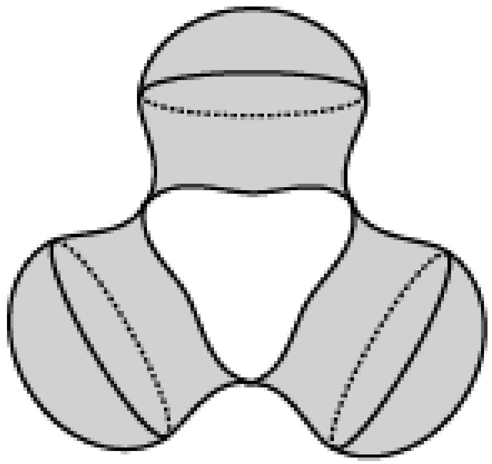
(1,1) critical point (l=1): $F(x,W) = W^3 - 3\mu x W - x^3 = 0$





(1,1) critical point (l=0): $F(x,W) = W^3 - 3\mu x W - x^3 + \mu^3 = 0$





Conclusion of Z_k symmetric amplitudes [CISY '09]

1. We studied the genenral k-cut two-matrix models.
2. The critical point and its critical potential are identified.
3. We found a natural extension of Kostov's formula in the general k-cut matrix models, which is simply written with the Jacobi polynomial.
4. We quantitatively showed what exactly happens in the Large N dynamics of the multi-cut matrix models.

Z_k breaking cases: [CISY2 '09]

$$x \simeq \mathbf{P}(t;\partial) = a^{-\hat{p}/2}\mathcal{A}(l;e^{\partial_l}) = C_0^{\mathcal{A}}\,\partial^{\hat{p}} + C_1^{\mathcal{A}}(t)\,\partial^{\hat{p}-1} + \cdots + C_{\hat{p}}^{\mathcal{A}}(t)$$

$$W \simeq \mathbf{Q}(t;\partial) = a^{-\hat{q}/2}\mathcal{B}(l;e^{\partial_l}) = C_0^{\mathcal{B}}\,\partial^{\hat{q}} + C_1^{\mathcal{B}}(t)\,\partial^{\hat{q}-1} + \cdots + C_{\hat{q}}^{\mathcal{B}}(t)$$

$$[\mathbf{P},\mathbf{Q}] = g_s I_k$$

Solve this system in the week coupling limit

The coefficients $C_i^{\mathcal{A}}$ and $C_i^{\mathcal{B}}$ are *k x k matrices*

1. The 0-th order: $[\mathbf{P},\mathbf{Q}] = 0$ → Simultaneous diagonalization

$$\mathbf{P}(t;\partial) \longrightarrow t^{\frac{\hat{p}}{\hat{p}+\hat{q}-1}}\, P_p^{(j)}(z), \qquad \mathbf{Q}(t;\partial) \longrightarrow t^{\frac{\hat{q}}{\hat{p}+\hat{q}-1}}\, Q_q^{(j)}(z),$$

2. The next order: (eigenvalues: j=1, 2, … , k)

$$qP_p^{(j)\prime}(z)Q_q^{(j)}(z) - pQ_q^{(j)\prime}(z)P_p^{(j)}(z) = const$$

EZJ-DKK eq.

they are no longer polynomials in general

Z_k symmetry breaking cases: $C_0^{\mathcal{A}} = \Gamma, \quad C_0^{\mathcal{B}} = \Gamma$

Proposal of the general k-cut (p,q) solution: [CISY2 '09]

1. The COSH solution: $F(\zeta,Q) = T_{k\hat{p}}\left(Q/\mu^{\hat{q}/2\hat{p}}\right) - T_{k\hat{q}}\left(\zeta/\sqrt{\mu}\right) = 0$

$$P^{(j)}_{\hat{p}} = \sqrt{\mu}\cosh\left(\hat{p}\tau + 2\pi i\frac{j-1}{k}\right), \quad Q^{(j)}_{\hat{q}} = \mu^{\hat{q}/2\hat{p}}\cosh\left(\hat{q}\tau + 2\pi i\frac{j-1}{k}\right)$$

$$\det\left(\zeta - \mathbf{P}\right) = 0 \iff T_k(\zeta/\sqrt{\mu}) - T_{k\hat{p}}(z) = 0$$

2. The SINH solution: $2 \mid k \wedge 2 \nmid k(\hat{q}-\hat{p})/2$

$$F(\zeta,Q) = (-1)^{\frac{k\hat{p}}{2}} T_{k\hat{p}}\left(-iQ/\mu^{\hat{q}/2\hat{p}}\right) - (-1)^{\frac{k\hat{q}}{2}} T_{k\hat{q}}\left(-i\zeta/\sqrt{\mu}\right) = 0$$

$$P^{(j)}_{\hat{p}} = \sqrt{\mu}\sinh\left(\hat{p}\tau + 2\pi i\frac{j-1}{k}\right), \quad Q^{(j)}_{\hat{q}} = \mu^{\hat{q}/2\hat{p}}\sinh\left(\hat{q}\tau + 2\pi i\frac{j-1}{k}\right)$$

$$\det\left(\zeta - \mathbf{P}\right) = 0 \iff (-1)^{\frac{k}{2}} T_k(-i\zeta/\sqrt{\mu}) - T_{k\hat{p}}(z) = 0$$

Only these solutions can satisfy the REALITY constraints

The matrix realization for the general (p,q) solutions

[CISY2 '09]

$$\mathbf{Q}^{(\hat{q})} = T_{\hat{q}}(z)\,\Gamma + \Big(T_{\hat{q}-1}(z) + T_{\hat{q}-3}(z) + \cdots\Big)M + \Big(T_{\hat{q}-2}(z) + T_{\hat{q}-4}(z) + \cdots\Big)(\Gamma - \Gamma^{-1})$$

$$[M, \Gamma + \Gamma^{-1}] = 0$$

In this formula we define: $T_0(z) \equiv 1/2$

Repeatedly using the addition formulae, we can show

$$[\mathbf{Q}^{(\hat{p})}, \mathbf{Q}^{(\hat{q})}] = 0$$

$$M_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & -1 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} \frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{2} - \frac{3}{2\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{2} - \frac{3}{2\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{2} - \frac{3}{2\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Conclusion of Z_k breaking cases:

[CISY2 '09]

1. We have proposed the solution to the general k-cut (p,q) Z_k breaking critical points.

2. They are *cosh/sinh solutions*. The *3,4,5-cut cases* are *proved* (the matrix realization).

3. Only the *cosh/sinh solutions* have the characteristic equations which are *algebraic equations with real coefficients*. (The other phase shifts are not allowed.)

4. This includes ONE-cut and TWO-cut cases as the special cases. The *cosh/sinh* expression indicates the Modular S-matrix of the worldsheet CFT.

Some of Future Issues

1. What is “l” ? (S)

2. Their connections ? (S)

3. The algebraic equations ? (S)

4. The geometry for k-cut solution
   (S, A in progress)

5. The annulus amplitudes (S, A)

Obrigado !