The Coupling of Chern-Simons Theory to Topological Gravity

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Lisbon, IST, Matrix Models and Geometry, December 2-9, 2009

Based on:
C.I., arXiv:0905.4631, Nucl. Phys. B825 (2010) 366.

Outline

1. Chern-Simons Topological Structure
2. BV actions
3. Topological Anomalies
   - Geometry of the Ward Identity
4. Topological strings
   - Anomaly cancellation
   - Generalized topological string amplitudes

Gauge BRST

- The classical CS action:

$$\Gamma_{CS} = \int_{M_3} \mathrm{Tr}\Big[\frac{1}{2} A\, d\, A + \frac{1}{3} A^3\Big]$$

- BRST invariance:

$$S_0\, A = D\, c \qquad S_0\, c = c^2$$

Anti-fields

- BV action

$$\Gamma_{BV} = \Gamma_{CS}[A] - \int_{M_3} \mathrm{Tr}\Big[\big(S_0\, A\big)\, A^* + \big(S_0\, c\big)\, c^*\Big]$$

- Anti-fields A^* of ghost # -1 and c^* of ghost # -2

$$A^* = (A^*)^a_{\mu\nu}\, T^a\, dx^\mu\, dx^\nu \qquad c^* = (c^*)^a_{\mu\nu\rho}\, T^a\, dx^\mu\, dx^\nu\, dx^\rho$$

- Extend BRST to anti-fields

$$S_0\, A^* = F + [A^*, c] \qquad S_0\, c^* = D\, A^* + [c^*, c]$$

Metric (in)dependence

- CS classical action is topological, i.e. it is independent of the 3-dimensional background metric $g_{\mu\nu}$.
- The gauge-fixed action depends on $g_{\mu\nu}$. E.g., for Landau's gauge

$$\Gamma_{g.f.} = S_0 \int_{M_3} \sqrt{g}\,\mathrm{Tr}\left[b\,D^{\mu}\,A_{\mu}\right] d^3x$$

where b is the anti-ghost and D_{μ} is covariant under diffeos.

- The issue of topological anomalies is if quantum averages

$$Z[g_{\mu\nu}] = \mathrm{e}^{i\,F[g]} = \int [d\,\Phi]\mathrm{e}^{\frac{i\,k}{4\,\pi}\,(\Gamma_{CS}[A]+\Gamma_{g.f.})}\,O(\Phi)$$

do depend on $g_{\mu\nu}$ or not.

The trick

- Let α^i parametrize the gauge-fixing term:

$$\Gamma(\alpha^i) = \Gamma_{class} + S_0\, \chi(\alpha^i)$$

- Extend the action of the BRST operator to the α^i's

$$s \equiv S_0 + \beta^i \partial_{\alpha^i} \qquad s^2 = 0$$

- The extended gauge-fixed action is s-invariant action

$$\hat{\Gamma}(\alpha^i, \beta^i) \equiv \Gamma_0 + s\, \chi(\alpha^i)$$

The trick (cont.)

- The new partition function depends on anti-commuting β^i's

$$Z(\alpha^i,\beta^i) = \int e^{\frac{i}{\hbar}\hat{\Gamma}(\alpha^i,\beta^i)} =$$

$$= Z^{(0)}(\alpha^i) + \beta^i Z^{(1)}_i(\alpha^i) + \beta^i \beta^j Z^{(2)}_{ij}(\alpha^i) + \cdots$$

- The first term in the expansion is the original $\Gamma_0(\alpha)$

$$Z^{(0)}(\alpha^i) = \int e^{\frac{i}{\hbar}\Gamma_0(\alpha^i)}$$

- $Z(\alpha,\beta)$ satisfies, up to anomalies, the generalized BRST identity

$$s\,Z(\alpha^i,\beta^i) = 0$$

The trick (cont.)

- The extended BRST identity translates into

$$\beta^{i_1}\,\beta^{i_2}\,\ldots\beta^{i_k}\,\beta^{i_{k+1}}\partial_{i_1}\,Z^{(k)}_{i_2\cdots i_{k+1}}(\alpha^i)=0$$

- The first identity just states the α-independence of $Z^{(0)}(\alpha^i)$

$$\beta^i\frac{\partial}{\partial\alpha_i}\,Z^{(0)}(\alpha^i)=0$$

- The other identities say that the $Z^{(k)}(\alpha)$ are *closed* k-forms on the manifold parameterized by the α^i.

The trick (cont.)

- In standard quantum field theory introducing $Z(\alpha^i, \beta^i)$ is just a technical trick: $Z^{(k)}(\alpha^i)$ with $k > 0$ have usually no direct physical meaning. As a matter of fact, they typically vanish, due to ghost number conservation.
- The advantage of reformulating the (classical) gauge independence of $Z(\alpha^i)$ as a BRST identity for the extended $Z(\alpha^i, \beta^i)$ is that this permits the use of powerful cohomological methods to investigate the corresponding quantum anomalies.

The trick (cont.)

- Both in string theory and in topological field theories the situation is different since observables have generally non-vanishing positive ghost number.
- In string theory the parameters α^i are coordinates on the moduli space of Riemann surfaces and the total ghost number of the observables is such that the non-vanishing component of $Z(\alpha, \beta)$ is a top-form on the relevant moduli space.
- In the context of CS theory, the role of the parameters α^i is played by the background metric $g_{\mu\nu}(x)$. The BRST operator S_0 is the one encoding gauge invariance of the classical action.

- Therefore Witten (1992) proposed to extend the BRST operator for CS to the background metric

$$S_0\, g_{\mu\nu} = \psi_{\mu\nu} \qquad S_0\, \psi_{\mu\nu} = 0 \qquad (1)$$

where $\psi_{\mu\nu}$ is the topological fermionic gravitino field with ghost number $+1$.

- Topological anomalies are anomalies of the classical BRST identity

$$S_0 F[g] = 0$$

- The BRST rules (1) are those of topological gravity as originally defined by Witten (1988). In this sense to discuss topological invariance of CS we need to couple it to topological gravity.

Topological gravity in 3d

- After Witten's work it was understood, (Baulieu, Singer 1988; Ouvry, Stora, van Baal 1988) that the "naive" topological gravity BRST rules (1) must be slightly modified.
- One needs to introduce also both 3-d reparametrization ghosts ξ^μ of ghost number $+1$, and its super-partner, the commuting super-ghost γ^μ of ghost number $+2$

$$
\begin{aligned}
s\, g_{\mu\nu} &= \psi_{\mu\nu} - \mathcal{L}_\xi\, g_{\mu\nu} \\
s\, \psi_{\mu\nu} &= \quad -\mathcal{L}_\xi\, \psi_{\mu\nu} + \mathcal{L}_\gamma\, g_{\mu\nu} \\
s\, \xi^\mu &= \quad -\frac{1}{2}\mathcal{L}_\xi\, \xi^\mu + \gamma^\mu \\
s\, \gamma^\mu &= \quad -\mathcal{L}_\xi\, \gamma^\mu
\end{aligned}
$$

where $\mathcal{L}_\xi$ is the infinitesimal diffeo with parameter ξ^μ.

- Topological gravity observables are cohomology classes of s on the space functionals which do not contain ξ^μ. This is called the equivariant s-cohomology.
- The rationale for this is, in brief, as follows. As explained, the "partition function" $Z(g,\psi)$ is to be thought of as a (closed) form on the space of 3-d metrics Met_3, with $\psi_{\mu\nu}$ playing the role of the differential $\delta g_{\mu\nu}$. $Z(g,\psi)$ is invariant under diffeomorphisms. But one wants more: one would like $Z(g,\psi)$ to descend to the orbit space

$$\mathcal{M} = Met_3/Diff(M_3)$$

A diffeomorphism invariant *function* on Met_3, of course, does define a function on $\mathcal{M}$. An invariant *form*, in general, does not. Those that do, are called basic forms. The equivariant machinery is constructed in such a way to produce forms on Met_3 that are basic.

Coupling CS to topological gravity

Main observation

Once we switch from “naive” S_0 to the equivariant s in the gravitational sector we must do the same in the “matter” gauge sector:

$$s = S_0 - \mathcal{L}_\xi + \cdots \qquad \text{on} \quad c, A, A^*, c^*$$

and determine the dots in such a way that

$$s^2 = 0 \qquad \text{on all fields}$$

- We obtain in this way

$$
\begin{aligned}
s\,c &= c^2 - \mathcal{L}_\xi\, c - i_\gamma(A) \\
s\,A &= D\,c - \mathcal{L}_\xi\, A - i_\gamma(A^*) \\
s\,A^* &= F + [A^*, c] - \mathcal{L}_\xi\, A^* - i_\gamma(c^*) \\
s\,c^* &= D\,A^* + [c^*, c] - \mathcal{L}_\xi\, c^*
\end{aligned}
$$

- i_γ is the contraction with the vector field γ^μ:

$$
i_\gamma(A) \equiv \gamma^\mu\, A_\mu \qquad i_\gamma(A^*) = \gamma^\mu\, A^*_{\mu\nu}\, dx^\nu \qquad \text{etc}
$$

N=2 supersymmetry

- On the algebra generated by all fields but ξ^μ, it is useful to introduce the operator

$$S \equiv s + \mathcal{L}_\xi$$

- $s^2 = 0 \Leftrightarrow S^2 = \mathcal{L}_\gamma \qquad \text{on all fields but } \xi^\mu$
- S is nilpotent when acting on reparametrization invariants functionals which are independent of ξ^μ.
- Such functionals are called equivariant. Observables must be equivariant.

- S can be decomposed as

$$S = S_0 + G_\gamma$$

where S_0 is Witten's original "naive" BRST and G_γ is:

$$G_\gamma\, c = i_\gamma(A) \qquad G_\gamma\, A = i_\gamma(A^*)$$
$$G_\gamma\, A^* = i_\gamma(c^*) \qquad G_\gamma\, c^* = 0$$
$$G_\gamma\, g_{\mu\nu} = 0 \qquad G_\gamma\, \psi_{\mu\nu} = \mathcal{L}_\gamma\, g_{\mu\nu} \qquad G_\gamma\, \gamma^\mu = 0$$

- S_0 and G_γ generate a supersymmetric algebra

$$S_0^2 = 0 \qquad G_\gamma^2 = 0 \qquad \{S_0, G_\gamma\} = \mathcal{L}_\gamma$$

BV action

- BV action associated to “naive” S_0 is linear in anti-fields

$$\Gamma_0 = \Gamma_{CS}[A] - \int_{M_3} \mathrm{Tr}\Big[(S_0\, A)\, A^* + (S_0\, c)\, c^* + (S_0\, g_{\mu\nu})\, (g^*)^{\mu\nu} \Big]$$

- For the equivariant s one starts from the same expression

$$\Gamma_{BV} = \Gamma_{CS} - \sum_{\Phi} \int_{M_3} (s\, \Phi)\, \Phi^* + \cdots$$

- However, since $s\, A$ is anti-field dependent

$$s\, A = \ldots - i_\gamma (A^*)$$

terms quadratic in the anti-fields are needed.

The complete BV action

$$\Gamma_{BV} = \Gamma_{CS} - \int_{M_3} \mathrm{Tr}\Big[\sum_{\Phi} (s\,\Phi)\,\Phi^* - \frac{1}{2}\, i_\gamma(A^*)\,A^*\Big]$$

- This action reproduces the equivariant BRST rules

$$s\,\Phi = (\Gamma_{BV}, \Phi) = -\frac{\delta^R\,\Gamma_{BV}}{\delta\,\Phi^*} \quad s\,\Phi^* = (\Gamma_{BV}, \Phi^*) = \frac{\delta^R\,\Gamma_{BV}}{\delta\,\Phi}$$

and the BV classical master equation

$$s\,\Gamma_{BV} = (\Gamma_{BV}, \Gamma_{BV}) = 0$$

Gauge-fixed action

- The gauge-fixed action Γ_χ depends on the functional $\chi[\Phi]$

$$\Gamma_\chi[\Phi] = \Gamma_{BV}[\Phi, \Phi^* = \frac{\delta\,\chi}{\delta\,\Phi}]$$

- The gauge-fixed BRST operator is

$$s_\chi\,\Phi = \big(s\,\Phi\big)\Big|_{\Phi^* = \frac{\delta\,\chi}{\delta\,\Phi}}$$

- Since $\Gamma_{BV}[\Phi, \Phi^*]$ not linear in the anti-fields

$$s_\chi^2\,\Phi = \sum_{\Phi'} \frac{\delta^R\,\Gamma_{BV}}{\delta\,\Phi'}\,\frac{\delta\,(s\,\Phi)}{\delta\,(\Phi')^*}\Big|_{\Phi^* = \frac{\delta\,\chi}{\delta\,\Phi}} = 0 \quad \text{only on shell}$$

- $s_\chi\,\Gamma_\chi = 0$

- Chern-Simons gauge-fixed action

$$\Gamma_\chi = \Gamma_{CS}[A] - s_\chi\left(\chi[\Phi]\right) + \frac{1}{2}\int_{M_3} \mathrm{Tr}\left(i_\gamma\left(\frac{\delta\chi}{\delta A}\right)\frac{\delta\chi}{\delta A}\right)$$

- Gauge-fixed BRST

$$s_\chi c = c^2 - \mathcal{L}_\xi c + i_\gamma(A)$$
$$s_\chi A = D c - \mathcal{L}_\xi A + i_\gamma\left(\frac{\delta\chi}{\delta A}\right)$$

- s_χ is nilpotent up to terms proportional to the e.o.m. of A:

$$s_\chi^2 = -\int_{M_3} \mathrm{Tr}\left[i_\gamma\left(\frac{\delta\Gamma_\chi}{\delta A}\right)\frac{\delta}{\delta A}\right]$$

- In conclusion, taking $\chi[\Phi]$ covariant under background diffeos and indipendent of ghost ξ^μ we obtain

Gauge-fixed CS action coupled to top. grav.

$$\Gamma_\chi = \Gamma_{CS}[A] - S_\chi\,(\chi[\Phi]) + \frac{1}{2}\int_{M_3} \mathrm{Tr}\left(i_\gamma\left(\frac{\delta\,\chi}{\delta\,A}\right)\frac{\delta\,\chi}{\delta\,A}\right)$$

which does not depend on ξ^μ.

Summary of first lecture

Action

$$\Gamma_\chi = \Gamma_{CS}[A] - S_\chi\left(\chi[\Phi]\right) + \frac{1}{2}\int_{M_3} \mathrm{Tr}\left(i_\gamma\left(\frac{\delta\chi}{\delta A}\right)\frac{\delta\chi}{\delta A}\right)$$

Off-shell BRST transformations

$$\begin{aligned}
s\,c &= c^2 - \mathcal{L}_\xi\, c - i_\gamma(A)\\
s\,A &= D\,c - \mathcal{L}_\xi\, A - i_\gamma(A^*)\\
s\,A^* &= F + [A^*, c] - \mathcal{L}_\xi\, A^* - i_\gamma(c^*)\\
s\,c^* &= D\,A^* + [c^*, c] - \mathcal{L}_\xi\, c^*
\end{aligned}$$

Summary of first lecture

On-shell BRST transformations

$$s_\chi\, c = c^2 - \mathcal{L}_\xi\, c + i_\gamma(A)$$
$$s_\chi\, A = D\, c - \mathcal{L}_\xi\, A + i_\gamma(\frac{\delta\, \chi}{\delta\, A})$$

Main surprise

To put CS on a curved manifold we must couple it not only to the metric $g_{\mu\nu}$ and its BRST partner $\psi_{\mu\nu}$ but also to the commuting vector field γ^μ.

CS global vector supersymmetry

- Coupling to classical background is one way to study the (quantum) properties of corresponding conserved currents of the theory in flat space.
- From the point of view of the global symmetries of the CS theory in flat space the necessity of the source γ^μ is related to the fact that the supercurrent associated to $\psi_{\mu\nu}$ is conserved only modulo S_0.

Skip next 6 frames

- Obviously

$$g_{\mu\nu} \leftrightarrow T_{\mu\nu} = \text{stress energy tensor}$$
$$\psi_{\mu\nu} \leftrightarrow S_{\mu\nu} = \text{super current}$$

- Since the metric enters the action only via gauge-fixing, we have

$$T_{\mu\nu} = S_0\, S_{\mu\nu} = S_0\, \frac{\delta \chi}{\delta\, g^{\mu\nu}}$$

- But what about γ^μ?
- The γ-dependent part of the action is

$$-G_\gamma\left(\chi[\Phi]\right) + \frac{1}{2}\int_{M_3} \mathrm{Tr}\left(i_\gamma\left(\frac{\delta\,\chi}{\delta\,A}\right)\frac{\delta\,\chi}{\delta\,A}\right) =$$

$$= -\int_{M_3} \mathrm{Tr}\left[\frac{\delta\chi}{\delta c}\, i_\gamma(A) + \frac{\delta\chi}{\delta\Lambda}\,\mathcal{L}_\gamma\, b + \right.$$

$$\left. -\frac{1}{2}\,\epsilon^{\mu\nu\rho}\,\gamma^\sigma\, g_{\rho\sigma}\frac{\delta\chi}{\delta A_\mu}\frac{\delta\chi}{\delta A_\nu}\right]$$

- Take for example Landau's gauge

$$\chi[\Phi] = \int_{M_3} \mathrm{Tr}\left[b\,\bar{D}^\dagger\,A\right] = \int_{M_3} \sqrt{g}\, b^{(0)}\,\bar{D}^\mu\,A_\mu\,d^3x$$

where b is the anti-ghost field.

- Then the γ-dependent part of the action becomes

$$-\frac{1}{2}\int_{M_3} \epsilon^{\mu\nu\rho}\,\gamma^{\sigma}\,g_{\rho\sigma}\,\mathrm{Tr}\,\partial_{\mu} b^{(0)}\,\partial_{\nu}\,b^{(0)} =$$
$$= \frac{1}{2}\int_{M_3} \det g\,\left(D^{\mu}\,\gamma^{\rho}\right)\,\epsilon_{\mu\nu\rho}\,\mathrm{Tr}\,b^{(0)}\,D^{\nu}\,b^{(0)}$$

- It vanishes for γ^{μ} *constant* ! The action becomes independent of γ^{μ} and $G_{\gamma} = \gamma^{\mu}\,G_{\mu}$ turns into a global vectorial supersymmetry

$$S_{\chi}\,c = c^2 - i_{\gamma}(A) \quad S_{\chi}\,A = D\,c - i_{\gamma}(\bar{D}^{\dagger}\,b)$$

(Delduc, Gieres, Sorella 1989)

- What are the corresponding conserved currents ?

- One might think that the conserved vectorial supercurrents are $S_{\mu\nu} = \frac{\delta \chi}{\delta g^{\mu\nu}}$. But this is not quite true.

$$\tilde{S}_{\mu\nu} = S_{\mu\nu} + \Delta_{\mu\nu} \qquad \partial^\mu \tilde{S}_{\mu\nu} = 0$$
$$S_{\mu\nu} = S_{\nu\mu} \qquad \Delta_{\mu\nu} = -\Delta_{\nu\mu}$$

- Neither $S_{\mu\nu}$ nor $\Delta_{\mu\nu}$ is conserved.
- $\Delta_{\mu\nu} = S_0\, J_{\mu\nu} \qquad J_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho}\, \mathrm{Tr}\, D^\rho\, b^{(0)}\, b^{(0)}$
- Consistent with conservation of the stress-energy tensor

$$T_{\mu\nu} = S_0\, \tilde{S}_{\mu\nu} = S_0\, S_{\mu\nu}$$
$$0 = \partial^\mu\, T_{\mu\nu} = S_0\, \partial^\mu\, \tilde{S}_{\mu\nu} = S_0\, \partial^\mu\, S_{\mu\nu}$$

- In conclusion the Noether procedure fails

$$h^{\mu\nu}\, T_{\mu\nu} + \psi^{\mu\nu}\, S_{\mu\nu} + \cdots$$
$$\delta h^{\mu\nu} = D^{(\mu}\, \xi^{\nu)} \qquad \delta\psi^{\mu\nu} = D^{(\mu}\, \gamma^{\nu)}$$

since the truly conserved current $\tilde{S}_{\mu\nu}$ is not symmetric. One needs another current

$$h^{\mu\nu}\, T_{\mu\nu} + \psi^{\mu\nu}\, S_{\mu\nu} + \phi^{\mu\nu}\, S_0\, J_{\mu\nu} + \cdots$$
$$\delta h^{\mu\nu} = D^{(\mu}\, \xi^{\nu)} \qquad \delta\psi^{\mu\nu} = D^{(\mu}\, \gamma^{\nu)} \quad \delta\phi^{\mu\nu} = D^{[\mu}\, \gamma^{\nu]}$$

- Working out Noether procedure in a way compatible with S_0-invariance one arrives to the extra coupling

$$D^{[\mu}\,\gamma^{\nu]}\,J_{\mu\nu} = \frac{1}{2}\big(D^{\mu}\,\gamma^{\rho}\big)\,\epsilon_{\mu\nu\rho}\,\mathrm{Tr}\,b^{(0)}\,D^{\nu}\,b^{(0)}$$

  that was found by coupling to topological gravity.

- In conclusion, from the point of view of the conserved currents of the theory in flat space the coupling to topological gravity encodes the relations

$$\partial^{\mu}\,T_{\mu\nu} = 0 \quad T_{\mu\nu} = S_0\,S_{\mu\nu}$$
$$\partial^{\mu}\,S_{\mu\nu} = -S_0\,\partial^{\mu}\,J_{\mu\nu}$$

Matter Observables

- Observables are S-cohomology classes modulo d.
- Collect fields and anti-fields of the gauge sector into a single *generalized form* with values in the Lie algebra of G:

$$\mathcal{A} \equiv c + A + A^* + c^*$$

- $\mathcal{A}$ is odd. It has total fermionic degree (ghost+form) $f = 1$.

- $\delta \equiv S - d + i_\gamma \Rightarrow \delta^2 = 0$ since

$$\delta^2 = S^2 - \{d, i_\gamma\} = \mathcal{L}_\gamma - \{d, i_\gamma\} = 0 \quad \text{on generalized forms}$$

- BRST transformations are

$$\delta\,\mathcal{A} = \mathcal{A}^2$$

- δ-classes $\Leftrightarrow$ S-classes modulo d, i.e. local observables.

- Since $\left(G_\gamma + i_\gamma\right)\mathcal{A} = 0$,

$$\delta\,\mathcal{A} = \delta_0\,\mathcal{A}$$

  where $\delta_0 \equiv S_0 - d$ is the “flat-space” BRST operator.
- Let $\tau_{a_1,\ldots a_m}$ be G-invariant completely anti-symmetric tensor with m indices a_i in the adjoint of G.
- For each $\tau_{a_1,\ldots a_m}$ one can construct the δ-cohomology class:

$$\langle \mathcal{A}^m \rangle \equiv \tau_{a_1,\ldots a_m}\,\mathcal{A}^{a_1}\ldots\mathcal{A}^{a_m}$$

- The smallest m is $m = 3$, for which there is a unique $\tau_{abc} = f_{abc}$:

$$\Omega_3 \equiv \frac{1}{3}\mathrm{Tr}\mathcal{A}^3 \equiv \Omega_3^{(0)} + \Omega_3^{(1)} + \Omega_3^{(2)} + \Omega_3^{(3)}$$

$$\Omega_3^{(0)} = \frac{1}{3}\,\mathrm{Tr}\,c^3$$

$$\Omega_3^{(1)} = \mathrm{Tr}\,A\,c^2$$

$$\Omega_3^{(2)} = \mathrm{Tr}\big[A^*\,c^2 + A^2\,c\,\big]$$

$$\Omega_3^{(3)} = \frac{1}{3}\,\mathrm{Tr}\big[A^3 + 3\,c^*\,c^2 + 3\,\{A, c\}\,A^*\big]$$

- Since $\Omega_3^{(3)}$ is S-invariant modulo d it is possible to add it to the BV action

$$\tilde{\Gamma}_{BV} = \Gamma_{BV} + t \int_{M_3} \Omega_3^{(3)}$$

- The new BV action generates deformed BRST transformation rules.
- The deformed action corresponds just to a rescaling of CS coupling constant.

$$\tilde{\Gamma}_{BV} = \frac{1}{(1+t)^2} \Gamma_{BV} \Big|_{\mathcal{A} \to (1+t)\mathcal{A}}$$

$$\delta_t \mathcal{A} = \{\tilde{\Gamma}_{BV}(t), \mathcal{A}\} = (1+t)\, \mathcal{A}^2$$

$$\delta_t^2 = 0$$

Higher-ghost deformations of CS

- Higher ghost observables depend on the gauge group G.
- For $G = SU(N)$ and $m = 3, 5, 7, \ldots, 2\,N - 1$

$$\tau^{(R)}_{a_1,\ldots a_m} = \mathrm{Tr}_R\, T^{[a_1} \cdots T^{a_m]} = d_m(R)\, \tau_{a_1,\ldots a_m}$$

  define single-trace invariants.
- All the other $SU(N)$ anti-symmetric tensors are multi-traces. They are obtained by multiplying and anti-symmetrizing the $N - 1$ “primitive” single trace invariants with $m = 3, 5, 7, \ldots, 2\,N - 1$.

- The corresponding 3-form of ghost number $m-3$

$$\Omega^{(3)}_{m-3} = \mathrm{Tr}\Big[c^*\, c^{m-1} + \frac{1}{2} A^* \big(A\, c^{m-2} + c\, A\, c^{m-3} + \\ + \cdots + c^{m-2}\, A\big) + \sum_{i,j \le m-3} A\, c^{m-i}\, A\, c^j\, A\, c^{i-j-3}\Big]$$

can be added to the BV action

$$\tilde{\Gamma}_{BV}(t_m) = \Gamma_{BV} + \sum_{i=1}^{N-2} t_i \int_{M_3} \Omega^{(3)}_{2\,i}$$

- The new BV action generates deformed nilpotent BRST

$$\delta_t \mathcal{A} = \{\tilde{\Gamma}_{BV}(t_i), \mathcal{A}\} = \mathcal{A}^2 + \sum_{i=1}^{N-2} t_i \, \mathcal{A}^{2\,i+2}$$

$$\delta_t^2 = 0$$

where

$$\left(\mathcal{A}^{m-1}\right)^a \equiv g^{aa_1} \, \tau_{a_1 a_2 \ldots a_m} \, \mathcal{A}^{a_2} \cdots \mathcal{A}^{a_m}$$

and g^{ab} is the invariant Killing metric of $su(N)$.

Effective action

- Quantum averages of matter observables

$$e^{i\,F[g_{\mu\nu},\psi_{\mu\nu},\gamma^\mu;t_i]} = \int \left[dA\,dc\,db\,d\,\Lambda\right] e^{i\,\frac{k}{2\,\pi}\,\tilde{\Gamma}_{BV}(t_i)\Big|_{\Phi^*=\frac{\delta\,\chi}{\delta\,\Phi}}}$$

- The classical BRST extended identity is

$$S\,F[g_{\mu\nu},\psi_{\mu\nu},\gamma^\mu;t_i] = \int_{M_3}\left[\frac{\delta\,F}{\delta g_{\mu\nu}}\,\psi_{\mu\nu} - \frac{\delta\,F}{\delta\psi_{\mu\nu}}\,\mathcal{L}_\gamma\,g_{\mu\nu}\right] = 0$$

- For $G = SU(N)$:

$$F[g,\psi,\gamma;t_i] = \sum_{n=0}^{\infty} F_{2n}[g,\psi,\gamma;t_i]$$

$$F_{2\,n}[g_{\mu\nu},\psi_{\mu\nu},\gamma^{\mu};t_i] = \sum_{\sum_\alpha i_\alpha = n} \mathcal{F}_{i_1;i_2;\ldots;i_\alpha;\ldots}[g,\psi,\gamma]\, t_{i_1}\cdots t_{i_\alpha}\cdots$$

- Ghost number conservation implies the selection rule

$$\sum_{\alpha} i_\alpha = n \qquad 1 \le i_\alpha \le N-1$$

For finite N there is only a finite number of terms in the sums above.

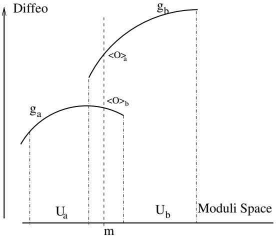
Geometrical interpretation of the Ward Identity

- Consider the space of 3-d metrics modulo diffeos:

$$\mathcal{M} = Met_3/Diff(M_3)$$

- $\psi_{\mu\nu}$ is the differential $\delta\, g_{\mu\nu}$ on Met_3. γ^{μ} is a 2-form on such a space.
- F_{2n} is a form of degree $2\, n$ on Met_3
- F_{2n} invariant under the action of diffeomorphisms of M_3(because of background covariance).

- $\{m^a\}$ local coordinates on $\mathcal{M}$.
- $g_{\mu\nu}(x; m^a)$ parametric family of 3-dimensional metrics which are diffeomorphic inequivalent for $\{m^a\} \neq \{(m^a)'\}$, i.e. a *local* gauge on Met_3.
- $d_m \equiv dm^a \frac{\partial}{\partial m^a}$ the exterior derivative on $\mathcal{M}$.



- When d_m acts on $g_{\mu\nu}(x;m)$ it does not give tensor covariant under m-dependent diffeos:

$$d_m g_{\mu\nu}(x,m) = \psi_{\mu\nu}(x;m) - \mathcal{L}_{\xi_m}\, g_{\mu\nu}(x;m^a)$$
$$d_m \psi_{\mu\nu}(x,m) = \mathcal{L}_{\gamma_m}\, g_{\mu\nu}(x;m) - \mathcal{L}_{\xi_m}\, \psi_{\mu\nu}(x;m)$$

- $\psi_{\mu\nu}(x,m)$ covariant under m-dependent diffeos
- $\xi^\mu(x;m)$ transforms as a connection under m-dependent diffeos
- $\gamma^\mu(x;m)$ covariant, is the curvature 2-form of the connection $\xi^\mu(x;m)$

$$d_m \xi_m + \frac{1}{2}\mathcal{L}_{\xi_m}\, \xi_m = \gamma_m$$

- d_m acts on $g_{\mu\nu}(x;m), \psi_{\mu\nu}(x;m), \xi^\mu(x;m)$ and $\gamma^\mu(x;m^a)$ in the same way as s acts on the topological gravity fields.
- Classical WI says that F_{2n} evaluated on $g_(x;m), \psi(x;m), \gamma(x;m)$ is a local closed $2n$-form on $\mathcal{M}$.
- Since $\psi(x;m)$ and $\gamma(x;m)$ transform covariantly under m-dependent diffeos — unlike $\xi^\mu(x;m)$ — this local closed form is globally defined on $\mathcal{M}$ if F_{2n} does not depend on ξ^μ
- Independence of ξ^μ is the equivariance condition.

- In this situation it would make sense to integrate the closed, globally defined form obtained by pulling back F_{2n} on the moduli space, on cycles $C_{2\,n}$ of moduli space $\mathcal{M}$ of dimension $2\,n$

$$I_{2n}(t_m) = \int_{C_{2\,n}} F_{2n}[g_{\mu\nu}(x;m), \psi_{\mu\nu}(x;m), \gamma^{\mu}(x;m); t_m]$$

- Were the classical Ward identity satisfied, the $I_{2n}(t_m)$ would be topological invariants of 3-dimensional variety M_3 depending on the homology class of $C_{2\,n}$. In the next section we will consider the possible quantum anomalies that can appear in the right hand side of the classical Ward identity.

Topological Anomalies

- Anomalous BRST identity for the $G = SU(N)$

$$S\, F_{2n}[g, \psi, \gamma] = \int_{M_3} A^{(3)}_{2\,n+1}[g, \psi, \gamma]$$

- The consistency condition for the anomaly is

$$S\, A^{(3)}_{2\,n+1} = d\, A^{(2)}_{2\,n+2}$$

where $A^{(3)}_{2\,n+1}$ and $A^{(2)}_{2\,n+2}$ are local.

- $A^{(3)}_{2\,n+1}$ belongs to a multiplet of anomaly forms

$$\mathcal{A}_{2\,n+4} = A^{(3)}_{2\,n+1} + A^{(2)}_{2\,n+2} + A^{(1)}_{2\,n+3} + A^{(0)}_{2\,n+4}$$

satisfying the descent equations

$$\delta\,\mathcal{A}_{2\,n+4} = 0$$

Local solutions of this equation modulo δ are local (equivariant) observables of 3-dimensional topological gravity.

Finding solution of the anomaly equations

- Define matrix-valued curvature 2-form

$$(R^{(2)})^{\mu}_{\ \nu} = \frac{1}{2}(R_{\alpha\beta})^{\mu}_{\ \nu}\, dx^{\alpha}\, dx^{\beta} = (d\,\Gamma + \Gamma^2)^{\mu}_{\ \nu}$$
$$(\Gamma)^{\mu}_{\ \nu} \equiv \Gamma^{\mu}_{\alpha\nu}\, dx^{\alpha}$$

and the generalized curvature:

$$\mathcal{R} \equiv R^{(2)} + R^{(1)} + R^{(0)}$$
$$(R^{(1)})^{\mu}_{\ \nu} = S\,\Gamma^{\mu}_{\ \nu} = \frac{1}{2}\Big[D\,\psi^{\mu}_{\ \nu} + D_{\nu}\,\psi^{\mu}_{\ \alpha}\, dx^{\alpha} - D^{\mu}\,\psi_{\alpha\nu}\, dx^{\alpha}\Big]$$
$$(R^{(0)})^{\mu}_{\ \nu} = D_{\nu}\,\gamma^{\mu}$$

- The main technical lemma is proving that the generalized curvature satisfies

$$\delta\,\mathcal{R} = 0$$

where:

$$\delta \equiv S - i_\gamma - D$$

is nilpotent *on reparametrization scalars*

$$\delta^2 = 0 \quad \text{on reparametrization scalars}$$

Solutions of the anomaly equations

$$\mathcal{A}_{4\,p} = \mathrm{tr}\,\mathcal{R}^{2\,p} \sim \left(\mathrm{tr}\,\mathcal{R}^{2}\right)^{p} \qquad p = 1, 2, \ldots$$

where the trace is taken over the Lorentz matrix indices.

- The corresponding anomaly has ghost number $4\,p-3$:

$$S\,F_{4\,(p-1)}[g,\psi,\gamma] = c_{2\,(p-1)}(t)\int_{M_3}\left(\mathrm{tr}\,\mathcal{R}^{2}\right)^{p} \qquad p = 1, 2, \ldots$$

- For $p = 1$ we recover the so-called framing anomaly (Witten 89):

$$A_1^{(3)} = \mathrm{tr}\,R^{(2)}\,R^{(1)} = \epsilon^{\mu\nu\rho}\,R^{\sigma}_{\rho}\,D_{\mu}\,\psi_{\nu\sigma}\,d^3\,x$$

- Anomalies with $p > 1$ involve the superghost γ^μ
- They are relevant for the higher-ghost deformations of CS theory with $t_i \neq 0$

The coefficients of the anomalies

- It is known that the coefficient of the $p = 1$ anomaly is non-zero
- For $G = SU(N)$ at 1-loop

$$c_0^{1-loop} = \frac{1}{12} \frac{N^2 - 1}{k}$$

- Witten proposed a formula to all orders, based on the Hamiltonian solution of the theory in terms of CFT:

$$c_0 = \frac{1}{12} \frac{N^2 - 1}{k + N}$$

- The coefficients of the anomalies A_{4p-3} for $p > 1$ are not known.
- The coefficients $c_{2\,(p-1)}$ for $p > 1$ are homogeneous polynomials of degree $2\,(p-1)$ in the t_i's, with t_i of weight i.

$$
\begin{aligned}
c_2(t_1, t_2) &= c_2^{(1)}\, t_2 + c_2^{(2)}\, t_1^2 \\
c_4(t_1, t_2, t_3, t_4) &= c_4^{(1)}\, t_4 + c_4^{(2)}\, t_2^2 + c_4^{(3)}\, t_2\, t_1^2 + \\
&\quad + c_4^{(4)}\, t_3\, t_1 + c_4^{(5)}\, t_1^4 \\
\dots & \quad \dots
\end{aligned}
$$

Topological strings anomaly inflow

- CS theory with $G = SU(N)$ on a 3-manifold M_3 describes a stack of N D-branes of (A-type) topological strings propagating on the non-compact CY 6-manifold $X_6 = T^*M_3$ (Witten 1992).
- Topological anomalies represents therefore the coupling of topological D-branes to the closed sector of topological strings.

- The target space field theory describing closed A-type topological strings propagating on X_6 is poorly understood.
- However anomaly inflow considerations require that there are anomalous couplings of the closed string theory which cancel CS topological anomalies.
- We can work out the form of such couplings.

- Physical vertex operators of the closed A topological model are de Rahm cohomology classes of forms on X_6 of degree 2 .
- The corresponding target space theory should involve a 2-form field $k^{(2)}$ whose linearized equations of motions are

$$d\,k^{(2)} = 0$$

- Gauge properties are described by a BRST operator S_0

$$S_0\,k^{(2)} = d\,k^{(1)} \qquad S_0\,k^{(1)} = d\,k^{(0)} \qquad S_0\,k^{(0)} = 0$$

- By repeating the reasoning that we applied to CS, one concludes that the closed target space theory should be coupled to 6-dimensional topological gravity. Introducing the BRST operator

$$s = S_0 - \mathcal{L}_\xi + \cdots = S - \mathcal{L}_\xi$$

where ξ^μ is an anti-commuting vector field on X_6, one arrives to the equivariant BRST transformations

$$\begin{aligned} S\, k^{(2)} &= d\, k^{(1)} \\ S\, k^{(1)} &= d\, k^{(0)} - i_\gamma(k^{(2)}) \\ S\, k^{(0)} &= -i_\gamma(k^{(1)}) \end{aligned}$$

where γ^μ is the 6-dimensional reparametrizations super-ghost.

- s so defined is nilpotent only up the e.o.m of $k^{(2)}$:

$$S^2 = \mathcal{L}_\gamma + i_\gamma (d\, k^{(2)}) \frac{\delta}{\delta\, k^{(2)}}$$

- We know a way to fix this: introduce forms with higher degrees (the anti-fields)

$$\mathcal{K} = k^{(0)} + k^{(1)} + k^{(2)} + k^{(3)} + k^{(4)} + k^{(5)} + k^{(6)}$$

- The nilpotent BRST trasformations are

$$\delta\, \mathcal{K} \equiv \big(S - d \ + i_\gamma \big)\, \mathcal{K} = 0$$

- Bershadsky & Sadov's (1994) target space theory for the A-model does involve forms $k^{(p)}$ with p other than 2.

- Topological D branes should act as sources for the closed string fields.
- We expect therefore that in presence of branes wrapped around M_3 the linearized e.o.m. of the closed string field get a source term

$$d\, k^{(2)} = \alpha\, \delta_{M_3}$$

- α is a constant proportional to the D-brane charge
- δ_{M_3} is a closed 3-form with support on the brane (the Poincaré dual of the cycle M_3).

- Correspondingly, the BRST transformations in presence of branes should also be modified

$$\delta\,\mathcal{K} = \alpha\,\delta_{M_3}$$

- Consistency requires

$$\delta^2\,\mathcal{K} = \alpha\,\delta\,\delta_{M_3} = \alpha\left(S - d + i_\gamma\right)\delta_{M_3} = 0$$

- $d\,\delta_{M_3} = 0$ since M_3 is closed.
- $i_\gamma\left(\delta_{M_3}\right) = 0$ since $\gamma^\mu\big|_{M_3} = 0$.
- $S\,\delta_{M_3} = 0$ since M_3 is a susy cycle.

Closed couplings cancelling brane topological anomalies

$$I_p = -\frac{c_{2\,(p-1)}(t)}{\alpha}\int_{X_6} \mathcal{A}_{4\,p}\,\mathcal{K} = -\frac{c_{2\,(p-1)}(t)}{\alpha}\int_{X_6} (\mathrm{tr}\mathcal{R}^2)^p\,\mathcal{K}$$

- Indeed:

$$\begin{aligned} S\,I_p &= -\frac{c_{2\,(p-1)}(t)}{\alpha}\int_{X_6} \mathcal{A}_{4\,p}\,\delta\,\mathcal{K} = \\ &= -c_{2\,(p-1)}(t)\int_{X_6} \mathcal{A}_{4\,p}\,\delta_{M_3} = \\ &= -c_{2\,(p-1)}(t)\int_{M_3} \mathcal{A}_{4\,p} = -S\,F_{4\,(p-1)} \end{aligned}$$

- For examples, the $p = 1$ anomaly is cancelled by the couplings

$$I_1 = \int_{X_6} \mathcal{K}\,\mathcal{A}_4 = \int_{X_6} \frac{1}{2}\Big[k^{(2)}\,\mathrm{tr}\,(R^{(2)})^2 + \\ + k^{(3)}\,\mathrm{tr}\,R^{(2)}\,R^{(1)} + k^{(4)}\,\mathrm{tr}\,\big((R^{(2)})\,R^{(0)} + \\ + (R^{(1)})^2\big) + k^{(5)}\,\mathrm{tr}\,R^{(1)}\,R^{(0)} + k^{(6)}\,\mathrm{tr}\,\big(R^{(0)}\big)^2\Big]$$

- Evidence confirming the presence in the closed theory of the coupling $k^{(2)}\,\mathrm{tr}\,(R^{(2)})^2$ of the right magnitude has been given by Gopakumar & Vafa, 1998.
- Anomalies with $p > 1$ provide in principle new tests of the duality between CS theory and the closed topological A-model.

Topological string interpretation of higher-ghost CS couplings

(Thanks to I. Melnikov for help in clarfying this part)

- Conventional open/closed string amplitudes are obtained by integrating a form of maximal degree d on the moduli space of Riemann surfaces $\mathcal{M}_{Riemann}$ (of given genus g, h boundary components, n_c closed punctures and n_o open punctures)

$$A(t) = \int_{\mathcal{M}_{Riemann}} \omega_0^{(d)}$$

- A(t) is a *function* of the *target space* moduli t which characterize the background topological non-linear sigma model.

- $\omega_0^{(d)}$ depends on the gauge-fixing term of the non-linear topological sigma model: It varies by an exact form on $\mathcal{M}_{Riemann}$ when one changes the gauge-fixing term, which for the A-model has the structure

$$\chi = G_{IJ}(\phi, \bar{\phi})\, \rho^I \star d\, \phi^J$$

where G_{IJ} is the target space CY metric.
- Let us therefore apply once again the “trick” to the 2d non-linear topological sigma model by extending the action of the sigma-model BRST operator to the background metric G_{IJ} (Witten 92)

$$s\, G_{IJ} = \psi_{IJ} + \cdots \qquad s\, \psi_{IJ} = 0 + \cdots$$

- Consequently $\omega_0^{(d)}$ is promoted to a generalized form

$$\Omega^{(d)} = \omega_0^{(d)} + \omega_1^{(d-1)} + \omega_2^{(d-2)} + \cdots \omega_d^{(0)}$$

where the lower index now denotes the form degree on the moduli space of 6-dimensional metrics G_{IJ}.

- The BRST identity reads

$$d_{Riemann}\Omega^{(d)} = d_{Target}\,\Omega^{(d)}$$

where $d_{Riemann}$ and d_{Target} are, respectively, the exterior differential on the moduli space of Riemann surfaces and target space metrics.

- Since in particular

$$d_{Target}\omega_0^{(d)} = d_{Riemann}\,\omega_1^{(d-1)}$$

integrated conventional string amplitudes do not change under a variation of the gauge-fixing term.

- Conventionally, the observables which intervene in the definition of $\omega_0^{(d)}$ are chosen in such a way that d coincides with the dimension d_{top} of the relevant moduli space of Riemann surfaces.
- For topological sigma models on CY complex 3-folds, this requirement selects among the possible observables those with standard *world-sheet* ghost number: this is either 2 for the closed or 1 for the open observables.
- For the A-model the world-sheet ghost number 2 observables correspond to closed 2-forms on the CY target space: these parametrize Kähler moduli of the target CY 3-fold.

- However, thanks to the "trick", we can now consider amplitudes involving observables with world-sheet ghost number *higher* than 2. In this case the form degree d would be *higher* than the dimension d_{top} of $\mathcal{M}_{Riemann}$.
- If $d = d_{top} + n$, the non-vanishing string amplitude

$$A_n(t) = \int_{\mathcal{M}_{Riemann}} \Omega^{(d)} = \int_{\mathcal{M}_{Riemann}} \omega_n^{(d-n)}$$

  is a n-form on the moduli space of the target metrics.
- The BRST identity will ensure that this form be closed and that its cohomology class be independent on the choice of the gauge-fixing term.

Generalized topological string amplitudes

- Conventional topological string amplitudes involving observables with *standard* ghost number are *functions* on the moduli space of target space metrics.
- BRST invariance ensures that such functions are independent of (some of) the metric data (i.e. topological).
- For the A-model this means independent of the complex structures of the target space.

Generalized topological string amplitudes

- Topological string amplitudes involving observables with *non-standard* ghost number must be thought of as *forms* on relevant moduli space of target space metrics. (For the A-model this would be the complex structure moduli space)
- BRST invariance ensures that such forms are *closed* and that their class is gauge-independent.
- Higher-ghost CS deformations describe topological open string amplitudes involving insertions in the bulk of closed string vertex operators with non-standard ghost number. The CS parameters t_i with $i > 0$ are functions of the *extended* moduli of the topological A-model.

Summary and conclusions

- CS on curved manifolds couples to the metric, the gravitino and the super-ghost field γ^μ. We derived the relevant classical BV action.
- The coupling to γ^μ is probed by higher ghost observables of CS theory. These observables are associated to higher-ghost deformations of CS.
- We computed the topological anomalies of the deformed CS which generalize the known framing anomaly.

Summary and conclusions (cont.)

- From the first-quantized point of view these CS deformations describe open topological string amplitudes in presence of closed vertex operators associated with the *thickened* moduli space of the A-model.
- We determined the couplings between the closed string fields and the gravitational background that cancel the topological anomalies of the open string sector.

Summary and conclusions

To do

Compute the coefficients of the higher ghost anomalies

$$
\begin{aligned}
c_0 &= \frac{N^2-1}{12}\,\frac{1}{k+N} \\
c_2(t_1,t_2) &= c_2^{(1)}\,t_2 + c_2^{(2)}\,t_1^2 \\
c_4(t_1,t_2,t_3,t_4) &= c_4^{(1)}\,t_4 + c_4^{(2)}\,t_2^2 + c_4^{(3)}\,t_2\,t_1^2 + \\
&\quad + c_4^{(4)}\,t_3\,t_1 + c_4^{(5)}\,t_1^4
\end{aligned}
$$

Summary and conclusions

Why?

- To compute new 3d invariants
- To test and understand topological open-closed string duality
- To understand closed topological string field theory
- To explore the “thickened" moduli space of topological A models.