Semester on Algebraic Geometry, IST, Lisbon
November 2009

Simple Hurwitz numbers

and

a proposition of Bouchard and Mariño

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arXiv : math-ph/0906.1206

with B. Eynard (CEA Saclay), M. Mulase (UC Davis), B. Safnuk (McMaster Univ.)

OUTLINE

1. What are (simple) Hurwitz numbers ?

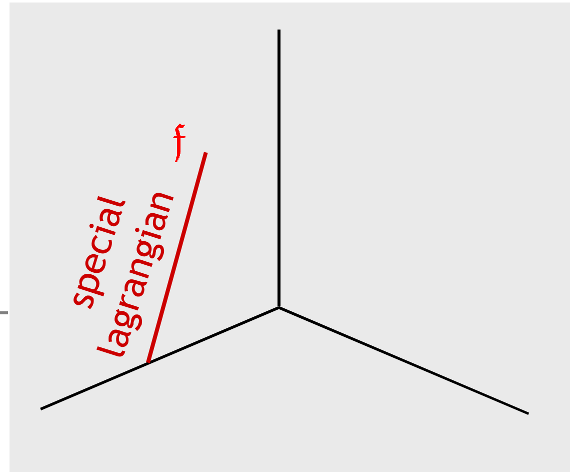
   Combinatorial problem

2. Topological string motivation

   General BKMP conjecture

   Application to 1-leg topo. vertex

   infinite framing $\rightarrow$ simple Hurwitz numbers



3. A matrix model for simple Hurwitz numbers

   Construction

   Application of the topological recursion

4. Coherent with other approaches (integrability & intersection theory)

Hurwitz numbers $h_{g,\mu}$ count classes (under topological equivalence) of

- Simple covers of $\mathbf{P}^1$, with

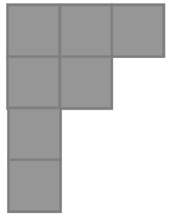
  genus g

  ramification profile μ over ∞

  $$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$$

  $r(g, \mu)$ simple branch points

  $$r(g, \mu) = 2g - 2 + |\mu| + \ell(\mu)$$

$\mu =$ 

$\ell(\mu)$ length

$|\mu|$ # boxes = # sheets

A cover may have discrete automorphisms among exchange of sheets.

$$h_{g,\mu} = \sum_{\pi} \frac{1}{|\mathrm{Aut}\ \pi|}$$

Hurwitz numbers $h_{g,\mu}$ count classes (under topological equivalence) of

- 'Branched graphs' Okounkov, Pandharipande (01)

  oriented surface of genus g built with

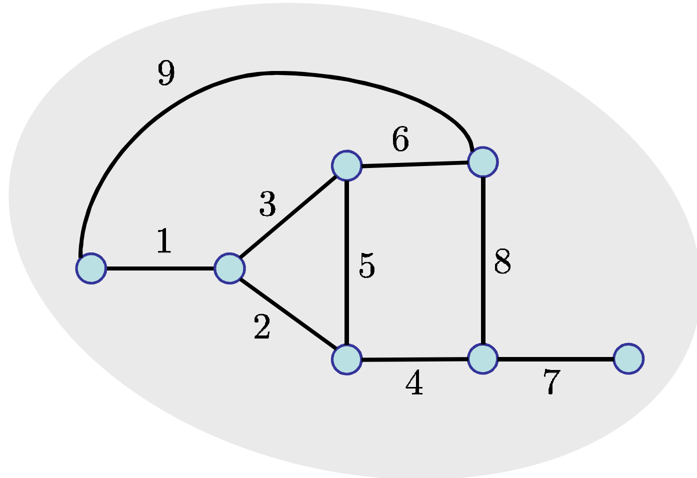
  $|\mu|$ vertices $\leftrightarrow$ sheets

  $r(g,\mu)$ edges* $\leftrightarrow$ simple branch points

  $\ell(\mu)$ faces $\leftrightarrow$ points projecting to ∞

  bijective edge labels respecting the orientation at each vertex

  there is a way of deducing $\mu \in \mathfrak{S}_{|\mu|}$

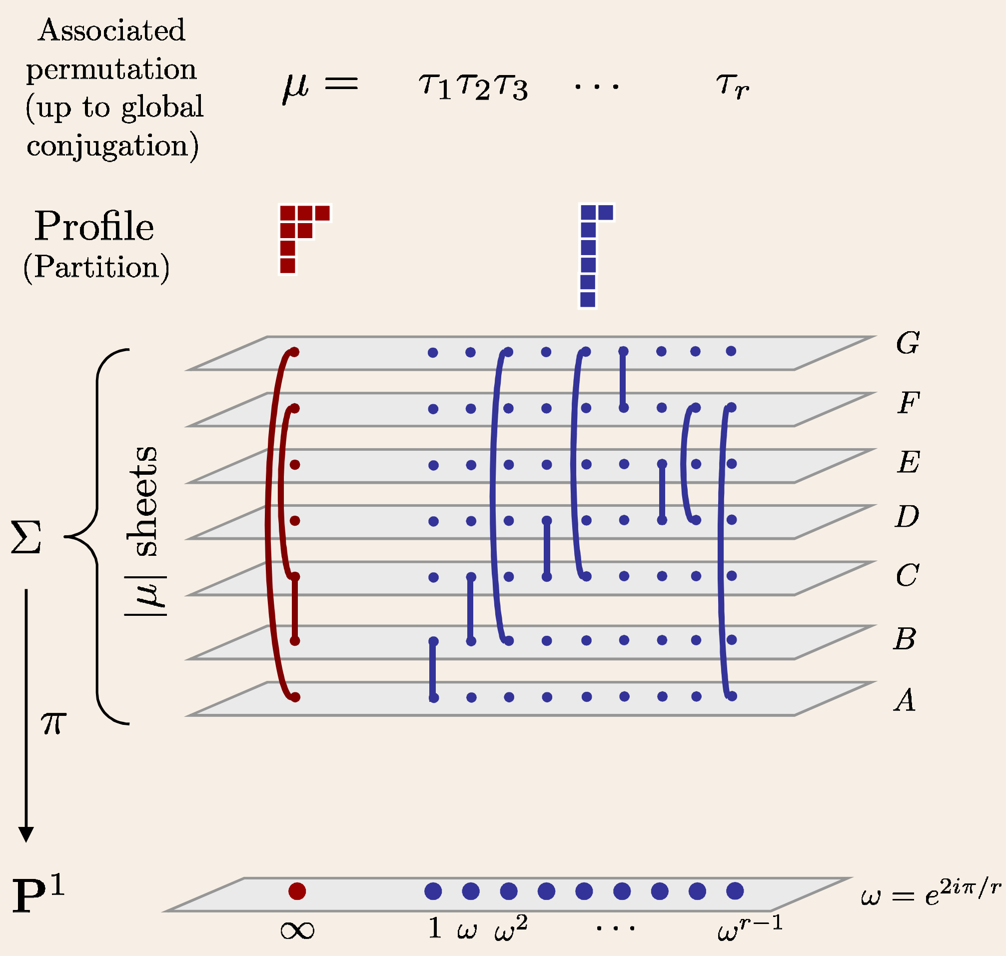


* No trivial edge

A branched graph may have automorphisms :
permutation of the vertices preserving the edge labels

$$h_{g,\mu} = \sum_{\mathcal{G}} \frac{1}{|\mathrm{Aut}\ \mathcal{G}|}$$

Simple cover π



Bijection compatible with automorphisms

Here, $|\mu| = 7$ and $|\mathrm{Aut}| = 1$

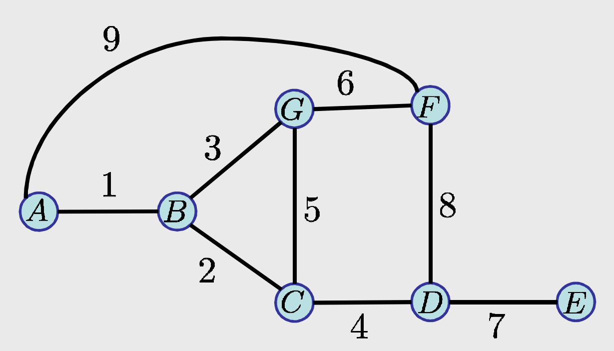
$r = 9$

$\ell(\mu) = 4$

$g = 0$

arbitrary labelling of vertices $\leftrightarrow$ sheets

Branched graph $\mathcal{G}$



Hurwitz numbers $h_{g,\mu}$ count also

- Topological classes of simple covers of $\mathbf{P}^1$
- Branched graphs

- Decomposition of μ in $r(g,\mu)$ transpositions
  (up to global conjugation)

$$\mu = \tau_1 \cdots \tau_r$$

$\rightsquigarrow$ Problem of enumeration in $\mathfrak{S}_{|\mu|}$: $\sum \frac{1}{|\mathrm{Aut}|}$

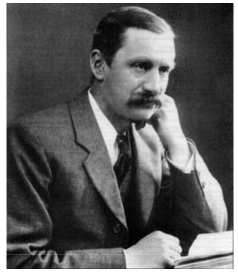
over decompositions $\mu = \gamma_1 \cdots \gamma_{r(g,\mu)}$
where γ_i has fixed conjugacy class C_i

Group theoretical formulation

Burnside formula (1898)

$$\begin{array}{c}\text{weigthed sum}\\ \text{over decompositions}\\ \mu = \gamma_1 \cdots \gamma_r\end{array} = \sum_{\substack{\lambda,\ \text{partition}\\ \text{of } |\mu| \text{ boxes}}} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2 f_\lambda(C_\mu) \prod_{i=1}^{r} f_\lambda(C_i)$$

where γ_i has fixed conjugacy class C_i



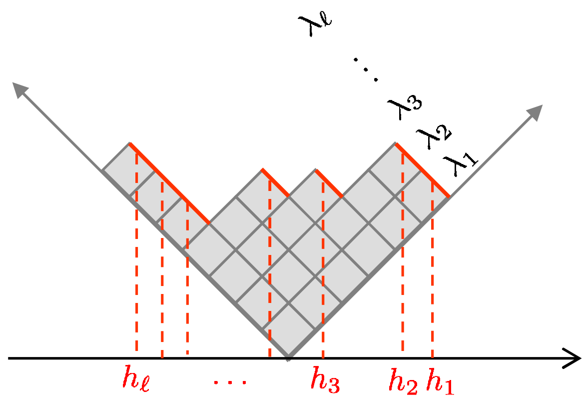
In the irreducible rep. of $\mathfrak{S}_{|\mu|}$ indexed by λ

$$f_\lambda(C) = \frac{|C|}{\dim \lambda} \chi_\lambda(C)$$

χ_λ = character

$$\frac{\dim \lambda}{|\lambda|!} = \frac{\Delta(h_1, \ldots, h_\ell)}{\prod_{i=1}^{\ell} h_i!}$$

Δ = Vandermonde determinant



'up' positions $h_i = \lambda_i - i + \ell$

Application of Burnside formula

$$\text{non connected } h_{g,\mu} = \sum_{\substack{\lambda,\text{ partition} \\ \text{of } |\mu| \text{ boxes}}} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2 f_\lambda(C_\mu)\, f_\lambda(C_{(2)})^{r(g,\mu)}$$

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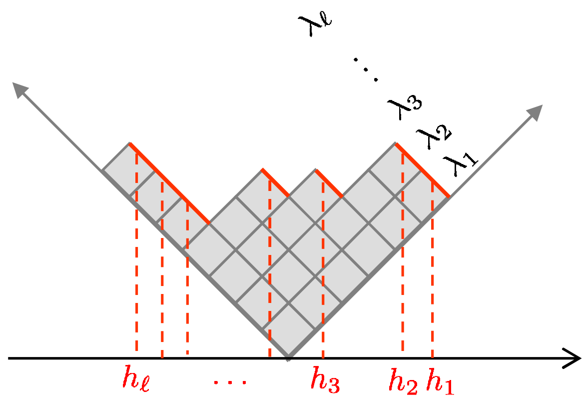
$C_{(2)}$ conjugacy class of a transposition



'up' positions $h_i = \lambda_i - i + \ell$

Combinatorial approach

1. First examples (« unstable covers »)

- $h_{0,\underbrace{\square\square\square\cdots\square\square}_{\mu_1}} = \mu_1^{\mu_1-3}$ enumeration of branched trees with μ_1 vertices ($r = \mu_1 - 1$ edges)

  $|\mathrm{Aut}| = 2$ if $\mu_1 = 2$

  $|\mathrm{Aut}| = 1$ else

- $h_{0,\begin{smallmatrix}\square\square\square\square\square & \mu_1 \\ \square\square\square\square & \mu_2\end{smallmatrix}} = \dfrac{(\mu_1+\mu_2)!}{\mu_1!\mu_2!}\dfrac{\mu_1^{\mu_1}\mu_2^{\mu_2}}{\mu_1+\mu_2}$ $(r = \mu_1 + \mu_2)$

Cayley's tree formula (1889)

$\mathcal{T}(n) = \{$trees with n vertices and bijective vertex labelling in $[1, n]\}$

$\nu(i)$ valence of the vertex i

$$\sum_{T \in \mathcal{T}(n)} \prod_{i=1}^{n} z_i^{\nu(i)} = z_1 \cdots z_n \, (z_1 + \cdots + z_n)^{n-2}$$

- Surjection $\mathcal{T}(\mu_1) \longrightarrow \begin{pmatrix} \text{branched trees} \\ \text{with } \mu_1 \text{ vertices} \end{pmatrix}$

  sort edges by lexicographic order induced by the vertices labels
  label the edges $1, \ldots, \mu_1 - 1$ in increasing order
  embed the tree in $\mathbf{P}^1$ such that edges are cyclically ordered at each vertex

$$h_{0,\underbrace{\square\square\square\cdots\square\square}_{\mu_1}} = \sharp \begin{pmatrix} \text{branched trees} \\ \text{with } \mu_1 \text{ vertices} \end{pmatrix} = \frac{|\mathcal{T}(\mu_1)|}{\mu_1} = \mu_1^{\mu_1 - 3}$$

Combinatorial approach

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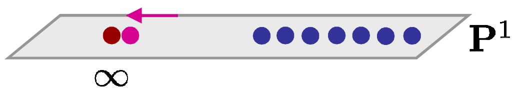
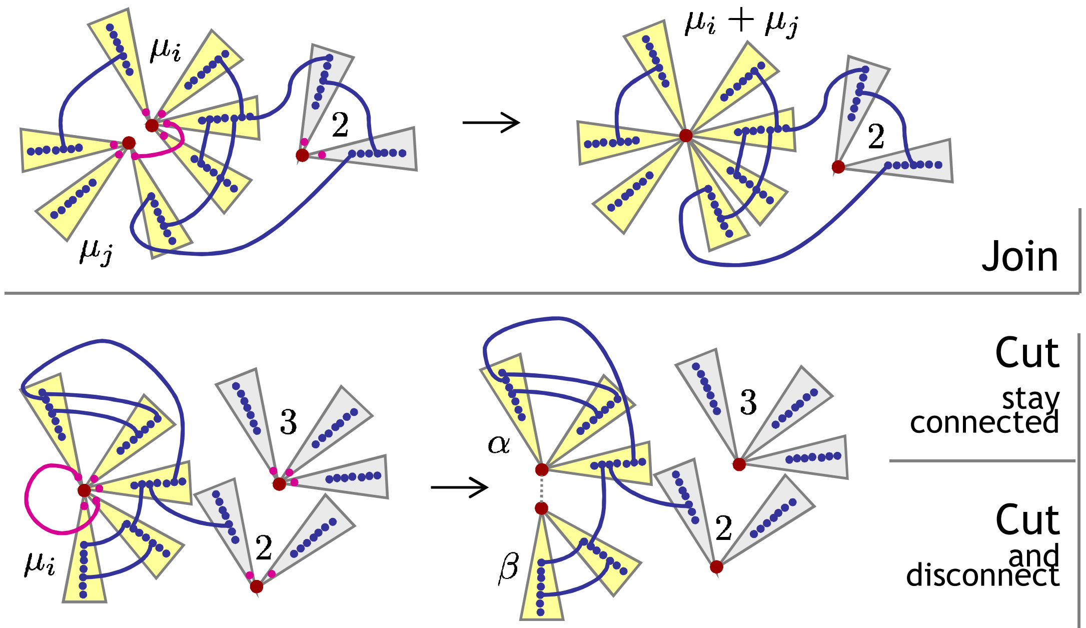
Combinatorial approach. We define $H_g(\mu) = |\mathrm{Aut}\ \mu| \cdot (\text{connected } h_{g,\mu})$

2. Recursion on $r(g,\mu)$: merge a branch point to ∞





Cut and join equation **_(1997)_**

Goulden, Jackson ; Vakil

$$r(g,\mu)H_g(\mu)$$
$$=$$
$$\sum_{i<j}(\mu_i+\mu_j)H_g(\mu \setminus \{\mu_i,\mu_j\} \sqcup (\mu_i+\mu_j))$$
$$+\ \frac{1}{2}\sum_{i=1}^{\ell(\mu)} \sum_{\alpha+\beta=\mu_i} \alpha\beta\ \Big[H_{g-1}(\mu \setminus \{\mu_i\} \sqcup \alpha, \beta)$$
$$+\sum_{\substack{g_a+g_b=g \\ \nu_a \sqcup \nu_b = \mu \setminus \{\mu_i\}}} H_{g_a}(\nu_a \sqcup \alpha)\, H_{g_b}(\nu_b \sqcup \beta)\Big]$$

(X, ω) Kähler space

There are two versions of topological string theory
with target space X

The closed perturbative amplitudes F_g are built with :

- Type A

Gromov-Witten invariants = intersection numbers in $\overline{\mathcal{M}_g}(\mathrm{X})$

= enumeration of genus g
complex curves drawn in X
wrt degree $\mathbf{D}$

$$F_g[\mathbf{t}] = \sum_{\mathbf{D}} N_{g|\mathbf{D}}^{\mathrm{X}}\, e^{-\mathbf{D}\cdot\mathbf{t}}$$

X ↔ *mirror symmetry* ↔ $\tilde{\mathrm{X}}$

- Type B

Periods matrix of ω

informations on the deformations
of the complex structure of X

$$t_I = \oint_{\mathcal{A}_I} \omega\,, \qquad \frac{\partial F_0}{\partial t_I} = \oint_{\mathcal{B}_I} \omega$$

$(\mathcal{A}_I, \mathcal{B}_I)_I$ symplectic basis of $H_3(\mathrm{X})$

For the open perturbative amplitudes $F_{g,k}[x_1,\dots,x_k]$:

- Type A — Open GW invariants

X

enumeration of genus g complex curve with k boundaries wrapped n_i times around some branes in X (framing $\mathfrak{f}\in\mathbf{Z}$)

mirror symmetry

$$F_{g,k}[\mathbf{t}|\mathbf{x},\mathfrak{f}] = \sum_{\mathbf{n}}\sum_{\mathbf{D}} N^{\mathrm{X}}_{g|\mathbf{D},\mathbf{n}}(\mathfrak{f})\, x_1^{n_1}\cdots x_k^{n_k} e^{-\mathbf{D}\cdot\mathbf{t}}$$

$\widetilde{\mathrm{X}}$

- Type B — Conjecture : 'Remodelling the B model' — Bouchard, Klemm, Mariño, Pasquetti (07)

for X, toric CY (branes = special lagrangians)

$$\omega_k^{(g)}[\mathbf{x},\mathfrak{f}] = \mathrm{d}_{x_1}\cdots\mathrm{d}_{x_k} F_{g,k}[\mathbf{x},\mathfrak{f}]$$

are given by topological recursion of matrix models with the mirror curve as spectral curve

Setting discovered for correlation functions of matrix models

developed by Eynard, Orantin (05)

- **Spectral curve** — data of $\mathcal{S} = (\mathcal{L}, x, y)$
  $\mathcal{L}$ Riemann surface
  $\mathrm{d}x$ meromorphic, y meromorphic
- **Branch points** — simple zeroes of $\mathrm{d}x$: $a_i \in \mathcal{L}$
- **Local involution** — $p \mapsto \overline{p} \in \mathcal{L}$ defined locally around a_i by $x(p) = x(\overline{p})$
- **Bergman kernel** — in case $\mathcal{L} = \mathbf{C}$, $B(p_1, p_2) = \dfrac{\mathrm{d}p_1\,\mathrm{d}p_2}{(p_1 - p_2)^2}$
  (else, known generalization)
- **Recursion kernel** — $K(p, p_1) = -\dfrac{1}{2}\dfrac{\int_{\overline{p}}^{p} B(\,\cdot\,, p_1)}{y(p)\,\mathrm{d}x(p)}$

- Local involution $p \mapsto \overline{p} \in \mathcal{L}$ defined locally around a_i by $x(p) = x(\overline{p})$
- Bergman kernel $B(p_1,p_2) = \frac{\mathrm{d}p_1\,\mathrm{d}p_2}{(p_1-p_2)^2}$ in case $\mathcal{L} = \mathbf{C}$, or its generalization
- Recursion kernel $K(p,p_1) = -\frac{1}{2}\frac{\int_{\overline{p}}^{p} B(\cdot\,,p_1)}{y(p)\,\mathrm{d}x(p)}$

- We define recursively meromorphic k-forms $\omega_k^{(g)}$

$$\omega_1^{(0)}(p) = 0 \qquad \omega_2^{(0)}(p_0,p_1) = B(p_0,p_1) - \frac{\mathrm{d}x(p_0)\,\mathrm{d}x(p_1)}{(x(p_0)-x(p_1))^2}$$

$$\omega_k^{(g)}(p_0,p_I) = \sum_i \operatorname*{Res}_{p\to a_i} K(p,p_0)\left[\text{"}\omega_{k+1}^{(g-1)}(p,\overline{p},p_I)\text{"} + {\sum_{J,h}}' \overline{\omega}_{|J|+1}^{(h)}(p,p_J)\overline{\omega}_{k-|J|}^{(g-h)}(\overline{p},p_{I\setminus J})\right]$$

where ${\sum}'_{J,h}$ ranges over $J \subseteq I, 0 \leq h \leq g$ but $(J,h) \neq (\emptyset,0),(I,g)$

- We define closed amplitudes F_g $(g \geq 2)$

$$F_g = \frac{1}{2-2g}\sum_i \operatorname*{Res}_{p\to a_i} \phi(p)\,\omega_1^{(g)}(p)$$ where $\mathrm{d}\phi = y\,\mathrm{d}x$, and more involved definition for F_0 and F_1

This formalism is called topological recursion

Many properties, among which :

- Symplectic invariance under $(x,y) \longrightarrow$

transformation	$F_g[\mathcal{S}]$	$\omega_k^{(g)}[\mathcal{S}]$
$(x, y+R(x))$	inv.	inv.
$(\lambda^{-1}x, \lambda y)$	inv.	inv.
$(-y, x)$	inv.	$(-1)^k$
(y, x)	inv.	$\star$

- Deformation of curves

when $y\,\mathrm{d}x \longrightarrow y\,\mathrm{d}x + \epsilon\,\Omega$

with the representation $\Omega(p_0) = -\int_{p\in\Omega^*} \Lambda_\Omega(p)\,B(p,p_0)$

$$\omega_k^{(g)}(p_K) \longrightarrow \omega_k^{(g)}(p_K) + \epsilon \int_{p\in\Omega^*} \Lambda_\Omega(p)\,\omega_{k+1}^{(g)}(p,p_K)$$

- Compatibility with limits

$$F_g[\lim\,\mathcal{S}] = \lim\,F_g[\mathcal{S}]$$

BKMP *conjecture* *(2007)*

Closed GW(X) are computed by $F_g[\mathcal{S}]$

Open GW(X) are computed by $\omega_k^{(g)}[\mathcal{S}]$

for X toric CY ⟷ *mirror*

$\widetilde{\mathrm{X}} : \mathrm{Pol}(e^x, e^y; t_\alpha) = uv$

$\mathcal{S} : \mathrm{Pol}(e^x, e^y; t_\alpha) = 0$ mirror curve

- Framing transformations $(x', y') = (x + \mathfrak{f}y, y) \quad \mathfrak{f} \in \mathbf{Z}$

These are the only symplectic transformations

∘ after which $\mathcal{S} : \mathrm{Pol}'(e^{x'}, e^{y'}) = 0$

∘ non trivial change of $\omega_k^{(g)}[\mathcal{S}]$ ⋆

Same closed amplitudes $F_g[\mathcal{S}]$

BKMP *conjecture* *(2007)*

Closed GW(X) are computed by $F_g[\mathcal{S}]$

Open GW(X) are computed by $\omega_k^{(g)}[\mathcal{S}]$

- Simplest case : $\mathrm{X} = \mathbf{C}^3$ with one brane

$$\mathcal{S} \; : \; 1 - e^x - e^y = 0 \qquad \text{std framing}$$

$$\mathcal{S}_{\mathfrak{f}} \; : \; e^{\mathfrak{f}y} - e^{(\mathfrak{f}+1)y} - e^x = 0 \qquad \text{after framing transformation}$$

$$\mathcal{S}_{\mathrm{Lambert}} \; : \; ye^{-y} = e^x \qquad \text{when } \mathfrak{f} \to \infty$$

Point of view of intersection theory :
open GW($\mathbf{C}^3$) for $\mathfrak{f} \to \infty$ are related to simple Hurwitz numbers

ELSV formula, see later

Prediction of* Bouchard *and* Mariño *(2008)

$$\omega_k^{(g)}[\mathcal{S}_{\mathrm{Lambert}}] = \mathrm{d}_{x_1} \cdots \mathrm{d}_{x_k} \sum_{\ell(\mu)=k} \frac{H_g(\mu)}{r(g,\mu)!} m_\mu(\mathbf{x})$$

$$m_\mu(\mathbf{x}) = \frac{1}{|\mathrm{Aut}\,\mu|} \sum_{\sigma \in \mathfrak{S}_k} \prod_{i=1}^{k} x_{\sigma(i)}^{\mu_i}$$

symmetric monomial basis

At least two different proofs :

- In fact, simple Hurwitz numbers comes from a matrix model
  one hermitian matrix + external field

with Eynard, Mulase, Safnuk (09)

Computing its spectral curve gives $\mathcal{S}_{\mathrm{Lambert}} \ : \ ye^{-y} = e^{x}$

- Topological recursion for $\mathcal{S}_{\mathfrak{f}} \ : \ e^{\mathfrak{f}y} - e^{(\mathfrak{f}+1)y} - e^{x} = 0$
  is equivalent to the cut-and-join equations for GW($\mathbf{C}^3$)

Zhou (09)

Unfolding the residue formula gives the cut-and-join equations

1 branch point, at $y(a) = \frac{\mathfrak{f}}{\mathfrak{f}+1}$

- In fact, simple Hurwitz numbers comes from a matrix model one hermitian matrix + external field

with Eynard, Mulase, Safnuk (09)

Computing its spectral curve gives $\mathcal{S}_{\mathrm{Lambert}} \;:\; ye^{-y} = e^{x}$

Back to Hurwitz numbers

Application of Burnside formula

$$\text{non connected } h_{g,\mu} = \sum_{\substack{\lambda,\ \text{partition} \\ \text{of } |\mu| \text{ boxes}}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 f_\lambda(C_\mu)\, f_\lambda(C_{(2)})^{r(g,\mu)}$$

In the irreducible rep. of $\mathfrak{S}_{|\mu|}$ indexed by λ

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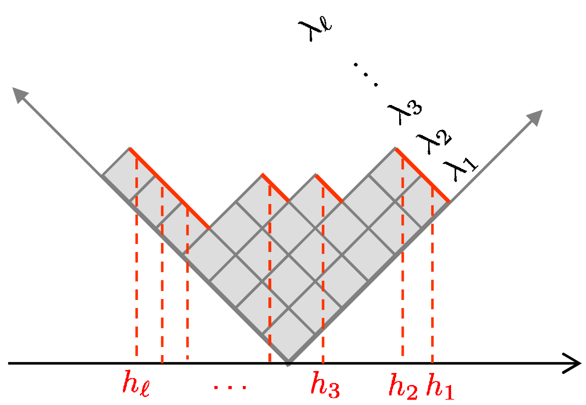
$$\frac{\dim\lambda}{|\lambda|!} = \frac{\Delta(h_1,\ldots,h_\ell)}{\prod_{i=1}^{\ell} h_i!}$$

Δ = Vandermonde determinant

$C_{(2)}$ conjugacy class of a transposition



'up' positions $h_i = \lambda_i - i + \ell$

Schur-Weyl duality

between reps. of $\mathfrak{S}_{|\mu|}$ *and reps. of* $\mathbf{U}(\ell(\mu))$

$$s_\lambda(\mathbf{v}) = \frac{1}{|\lambda|!} \sum_{|\mu|=|\lambda|} |C_\mu| \chi_\lambda(C_\mu)\, p_\mu(\mathbf{v})$$ Schur polynomials

$$p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}$$ where p_k is the k-th power sum

suggests to define the Hurwitz partition function as

$$Z(g_s, t|\mathbf{v}) = \sum_{\text{partition } \mu} \sum_{g=0}^{\infty} \frac{g_s^{2g-2}\, t^{|\mu|}}{r(g,\mu)!} g_s^{\ell(\mu)} p_\mu(\mathbf{v}) \cdot (\text{non connected } h_{g,\mu})$$

$$F(g_s, t|\mathbf{v}) = \ln Z(g_s, t|\mathbf{v})$$

$\mathbf{v} = (v_1, \ldots, v_N, \ldots)$ infinite formal variable

- We restrict this sum over partitions μ of size $\leq$ N to define Z_N

$$\mu_1 \geq \ldots \geq \mu_\ell > \mu_{\ell+1} = \ldots = \mu_N = 0$$

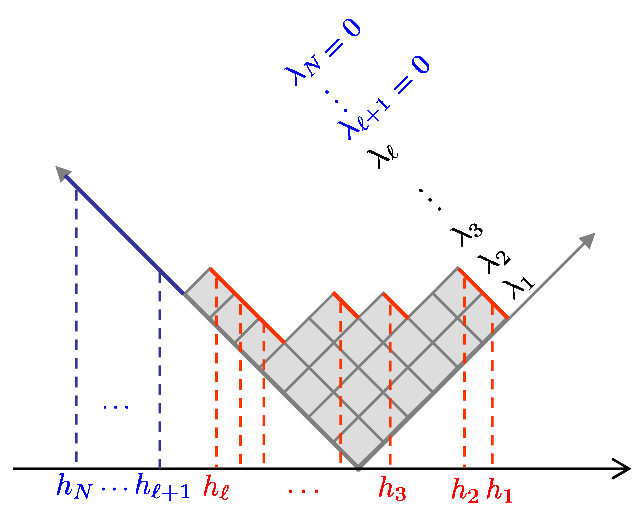
$$\Rightarrow \; Z_N(g_s, t|\mathbf{v}) = \sum_{\lambda_1 \geq \ldots \geq \lambda_N \geq 0} \frac{\dim \lambda}{|\lambda|!}\, s_\lambda\left(\frac{t\mathbf{v}}{g_s}\right)\, e^{g_s f_\lambda(C_{(2)})}$$

$\mathbf{v} = (v_1, \ldots, v_N)$ formal variable of size N

- It is a sum over extended partitions of size N

$$\frac{\dim \lambda}{|\lambda|!} = \frac{\Delta(h_1, \ldots, h_N)}{\prod_{i=1}^N h_i!}$$

consistent with addition of $0, \ldots, 0$ to $\lambda_1 \geq \ldots \geq \lambda_\ell > 0$



$$h_i = \lambda_i - i + N$$

- To extract $h_{g,\mu}$ for a given $|\mu| = n$ it is enough to take $N \geq n$

$$Z_N(g_s, t|\mathbf{v}) = \frac{1}{N!} \sum_{h_1,\dots,h_N \geq 0} \frac{\dim \lambda_{\mathbf{h}}}{|\lambda_{\mathbf{h}}|!} \, s_{\lambda_{\mathbf{h}}}\left(\frac{t\mathbf{v}}{g_s}\right) \, e^{g_s f_{\lambda_{\mathbf{h}}}(C_{(2)})}$$

- We take $h_1 > \cdots > h_N$ as variables, and symmetrize

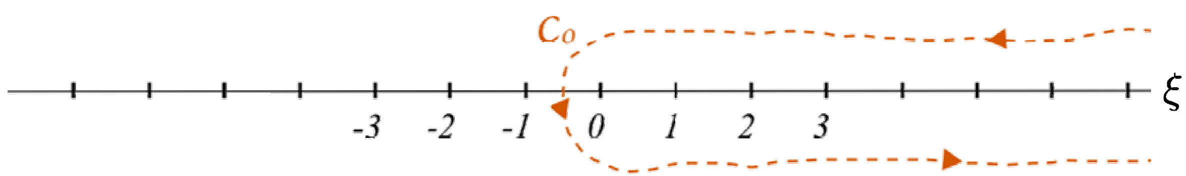
  $h_1, \dots, h_N$ defines a partition $\lambda_{\mathbf{h}}$

$$Z_N(g_s, t|\mathbf{v}) = \frac{1}{N!} \sum_{h_1,\dots,h_N \geq 0} \frac{\dim \lambda_\mathbf{h}}{|\lambda_\mathbf{h}|!} \, s_{\lambda_\mathbf{h}}\left(\frac{t\mathbf{v}}{g_s}\right) \, e^{g_s f_{\lambda_\mathbf{h}}(C_{(2)})}$$

1 • Turn the sums into contour integrals

$$\sum_{h_1,\dots,h_N \geq 0} \cdot \longrightarrow \frac{1}{2i\pi} \oint_{\mathcal{C}_0^N} \prod_{i=1}^{N} \mathrm{d}\xi_i \, F(\xi_i) \cdot \qquad F(\xi) = \frac{\pi e^{-i\pi\xi}}{\sin \pi\xi} = -\Gamma(\xi)\Gamma(\xi+1)e^{-i\pi\xi}$$

$$\frac{\dim \lambda_\mathbf{h}}{|\lambda|!} \longrightarrow \frac{\Delta(\xi)}{\prod_{i=1}^{N} \Gamma(\xi_i + 1)}$$



$$Z_N(g_s, t|\mathbf{v}) = \frac{1}{N!} \sum_{h_1,\ldots,h_N} \frac{\dim \lambda_{\mathbf{h}}}{|\lambda_{\mathbf{h}}|!} \, s_{\lambda_{\mathbf{h}}}\left(\frac{t\mathbf{v}}{g_s}\right) \, e^{g_s f_{\lambda_{\mathbf{h}}}(C_{(2)})}$$

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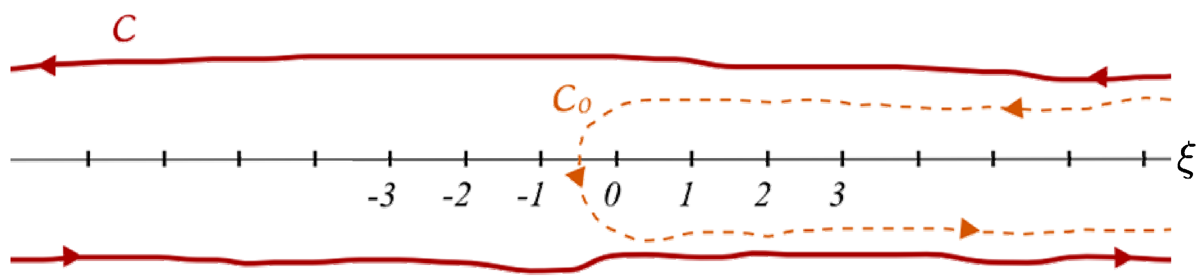
$$\frac{\dim \lambda_{\mathbf{h}}}{|\lambda|!} \longrightarrow \frac{\Delta(\xi)}{\prod_{i=1}^{N} \Gamma(\xi_i + 1)}$$



$$Z_N(g_s,t|\mathbf{v}) = \frac{1}{N!}\sum_{h_1,\dots,h_N\geq 0}\frac{\dim\lambda_{\mathbf{h}}}{|\lambda_{\mathbf{h}}|!}\, s_{\lambda_{\mathbf{h}}}\left(\frac{t\mathbf{v}}{g_s}\right)\, e^{g_s f_{\lambda_{\mathbf{h}}}(C_{(2)})}$$

2 • Write Schur functions in term of Itzykson-Zuber integral

$$\mathrm{I}(\mathbf{A},\mathbf{B}) = \int_{\mathbf{U}(N)} \mathrm{d}U\; e^{\mathrm{Tr}(\mathbf{A}U\mathbf{B}U^\dagger)} = \frac{\det\left(e^{A_iB_j}\right)}{\Delta(\mathbf{A})\,\Delta(\mathbf{B})}$$

$$\Downarrow$$

$$s_{\lambda_\xi}\left(\frac{t\mathbf{v}}{g_s}\right) = \frac{\det((tv_i/g_s)^{\xi_j})}{\Delta(t\mathbf{v}/g_s)} = \Delta(\xi)\,\frac{\Delta(\mathbf{R})}{\Delta(t\mathbf{v}/g_s)}\,\mathrm{I}(\xi,\mathbf{R}+\ln(t/g_s))$$

with $\mathbf{R} = \ln\mathbf{v}$

$$Z_N(g_s,t|\mathbf{v}) = \frac{1}{N!} \sum_{h_1,\dots,h_N \geq 0} \frac{\dim \lambda_{\mathbf{h}}}{|\lambda_{\mathbf{h}}|!} s_{\lambda_{\mathbf{h}}}\left(\frac{t\mathbf{v}}{g_s}\right) e^{g_s f_{\lambda_{\mathbf{h}}}(C_{(2)})}$$

3 • $f_{\lambda_{\mathbf{h}}}(C_{(2)})$ do not mix the h_i's

$$f_{\lambda_{\mathbf{h}}}(C_{(2)}) = \sum_{i=1}^{N} \frac{1}{2} h_i^2 - \left(N - \frac{1}{2}\right) h_i + \mathrm{cte}(N)$$

$$Z_N(g_s,t|\mathbf{v}) = \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\,\frac{e^{\mathrm{cte}(N)}}{(2i\pi)^N N!}\int_{\mathbf{U}(N)} \mathrm{d}U \oint_{\mathcal{C}^N} \Delta(\xi)^2 \prod_{j=1}^{N} \mathrm{d}\xi_j$$

$$\cdot \left[\prod_{j=1}^{N}\left(-e^{i\pi\xi_j}\Gamma(\xi_j)\right)\, e^{\frac{1}{2}g_s\xi_j^2-\left(N-\frac{1}{2}\right)g_s\xi_j+\ln(g_s/t)\xi_j}\right] e^{\mathrm{Tr}(\xi U\mathbf{R}U^\dagger)}$$

Matrix model for simple Hurwitz numbers

$$Z_N(g_s,t|\mathbf{v}) = \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\,\frac{e^{\mathrm{cte}(N)}}{(2i\pi)^N N!}\int_{H_N(\mathcal{C})} \mathrm{d}M\, e^{-\frac{1}{g_s}\mathrm{Tr}(V(M)-M\mathbf{R})}$$

- $H_N(\mathcal{C})$ set of normal matrices with eigenvalues in $\mathcal{C}$
- $\mathbf{R} = \ln \mathbf{v}$ external field (diagonal matrix)
- Potential $V(M) = -\frac{M^2}{2} + \left(N-\frac{1}{2}\right)g_s M + \ln(g_s/t)M + i\pi M - g_s \ln\left(\Gamma(-M/g_s)\right)$

- In these matrix models, the spectral curve $\mathcal{S}$ is characterized by

$$\begin{cases} y(x) = \widetilde{W}_1^{(0)}(x) - V'(x) \\ \widetilde{W}_1^{(0)}(x) = P_1^{(0)}(x, y(x)) \end{cases}$$

defined as a power series in g_s and t

where $\widetilde{W}_1^{(0)}(x) = \left\langle \mathrm{Tr}\, \frac{1}{x - M} \right\rangle^{(0)}$ and $P_1^{(0)}(x,y) = \left\langle \mathrm{Tr}\, \frac{V'(x) - V'(M)}{x - M} \frac{1}{y - \mathbf{R}} \right\rangle^{(0)}$

- We define the correlation functions

$$(k > 0) \quad \widetilde{W}_k(x_1, \cdots, x_k) = \left\langle \prod_{i=1}^{k} \mathrm{Tr}\, \frac{1}{x_i - M} \right\rangle_c$$

c = cumulants

$$\widetilde{W}_0 = F = \ln Z$$

- In these matrix models, the spectral curve $\mathcal{S}$ is characterized by

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- Recall the definition

$$Z_N(g_s, t|\mathbf{v}) = \sum_{\substack{\text{partition } \mu \\ |\mu| \leq N}} \sum_{g=0}^{\infty} \frac{g_s^{2g-2}\, t^{|\mu|}}{r(g,\mu)!} g_s^{\ell(\mu)} p_\mu(\mathbf{v}) \cdot (\text{non connected } h_{g,\mu})$$

$\Rightarrow$ Z_N only depends on the power sums $\widetilde{p}_m = g_s \sum_{i=1}^{N} v_i^m \qquad (0 \leq m \leq N)$

- In these matrix models, the spectral curve $\mathcal{S}$ is characterized by

$$\begin{cases} y(x) = \widetilde{W}_1^{(0)}(x) - V'(x) \\ \widetilde{W}_1^{(0)}(x) = P_1^{(0)}(x, y(x)) \end{cases}$$

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- We define the correlation functions

$$\widetilde{W}_k(x_1,\cdots,x_k) = \left\langle \prod_{i=1}^{k} \mathrm{Tr}\,\frac{1}{x_i - M} \right\rangle_c \qquad W_0 = F = \ln Z$$

- I state that there is a well-defined 'topological expansion'

$$\widetilde{W}_k^{(g)}(x_1,\cdots,x_k) = \sum_{g=0}^{\infty} g_s^{2g-2+k}\,\widetilde{W}_k^{(g)}(x_1,\ldots,x_k) \qquad (k \in \mathbf{N})$$

and that $\omega_k^{(g)}[\mathcal{S}] = \widetilde{W}_k^{(g)}(x_1,\ldots,x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k$ if $(k,g) \neq (1,0)$

- Solution for the spectral curve

$$\mathcal{S}(g_s,t,\mathbf{v}) \ : \ \begin{cases} x(z) &= z + g_s \sum_{i=1}^N \frac{1}{y_i(z-z_i)} + \frac{1}{y_i z_i} \\ y(z) &= -z + \ln(z/t) + c_0 + \frac{c_1}{z} + O(g_s^2) \end{cases}$$

with $y(z_i) = \ln v_i$

$y'(z_i) = y_i,\ c_0,\ c_1,\ O(g_s^2)$ fixed by consistency relations

- One can check that order by order

$x \mapsto y(x)$ only depends on the power sums $\tilde{p}_m = g_s \sum_{i=1}^N v_i^m$

- For $g_s = 0$, it is a Lambert curve $\mathcal{S}(0,t) \ : \ y(x) = -x + \ln(x/t)$

- We find the curve of Bouchard-Mariño $\mathcal{S}_{\mathrm{Lambert}} \ : \ e^x = y e^{-y}$

  if we set $t = 1$

  if we exchange x and y

We apply the properties of the topological recursion

- By symplectic invariance

$F_g[\mathcal{S}_{x\leftrightarrow y}(g_s,t,\mathbf{v})]$ is the generating function of connected simple Hurwitz numbers

advantage of $x \leftrightarrow y$: only 1 branch point

- BM generating function is obtained by studying variations of v_i

$$\partial_{v_1}\cdots\partial_{v_k} \sum_{\ell(\mu)=k} \frac{H_g(\mu)}{r(g,\mu)!}\, t^{|\mu|}\, m_\mu(v_1,\ldots,v_k) = \lim_{g_s\to 0} \frac{\partial^k F_g[\mathcal{S}_{x\leftrightarrow y}(g_s,t,\mathbf{v})]}{\partial(g_s\ln v_1)\cdots\partial(g_s \ln v_k)}$$

when $v_{k+1} = \ldots = v_N = 0$

- To compute variations wrt $g_s \ln v_i$: general principles

if $x\,\mathrm{d}y \longrightarrow x\,\mathrm{d}y + (g_s \delta \ln v_i)\,\Omega_i$ with $\Omega_i(z_0) = -\int_{z\in\Omega_i^*} \Lambda_i(z)\,\frac{\mathrm{d}z\,\mathrm{d}z_0}{(z-z_0)^2}$

then $$\frac{\delta\left(\omega_k^{(g)}(z_I)\right)}{\delta(g_s \ln v_i)} = \int_{z\in\Omega_i^*} \Lambda_i(z)\,\omega_{k+1}^{(g)}(z, z_I)$$

- To compute variations wrt $g_s \ln v_i$: we find

if $x\,\mathrm{d}y \longrightarrow x\,\mathrm{d}y + (g_s \delta \ln v_i)\,\Omega_i$ with $\Omega_i(z_0) = -\operatorname*{Res}_{z \to z_i} \frac{1}{z - z_i} \frac{1}{y'(z)} \frac{\mathrm{d}z\,\mathrm{d}z_0}{(z - z_0)^2}$

so, $\frac{\delta\left(\omega_k^{(g)}(z_K)\right)}{\delta(g_s \ln v_i)} = \frac{\omega_{k+1}^{(g)}(z_i, z_K)}{\mathrm{d}y(z_i)}$ (recall $y(z_i) = \ln v_i$)

- Compatibility with limits :

$$\partial_{v_1} \cdots \partial_{v_k} \sum_{\ell(\mu)=k} \frac{H_g(\mu)}{r(g,\mu)!}\, t^{|\mu|}\, m_\mu(v_1, \ldots, v_k) = \frac{\omega_k^{(g)}[\mathcal{S}_{x\leftrightarrow y}(0,t)](z_1, \ldots, z_k)}{\mathrm{d}y(z_1) \cdots \mathrm{d}y(z_k)}$$

$\square$

- The spectral curve $\mathcal{S}_{x\leftrightarrow y}(g_s, t, \mathbf{v})$ is a special case of

$$\mathcal{S}_K(t_3, \dots, t_m, \dots) \ : \ \begin{cases} x(\zeta) &= -1 - \ln t - \frac{1}{2}\zeta^2 \\ y(\zeta) &= 1 - 2\zeta + \sum_{m\geq 1} t_{m+2}\zeta^m \end{cases}$$

up to notations,
the Kontsevich curve
Kontsevich (92)

Here : $t_3 = 3,\ t_4 = \frac{1}{3},\ t_{m+1} = \frac{t_m}{m} - \frac{1}{2}\sum_{l=2}^{m-2} t_{l+2}t_{m+2-l}$

The Lambert function performs the resummation

- A few properties :

$Z[\mathcal{S}_K(t_3, \dots, t_m, \dots)]$ is a τ function of the KP hierarchy

in agreement with works of
Kazarian (08) ; Mironov, Morozov (09)

- A few properties : in the intersection theory of $\overline{\mathcal{M}_{g,k}}$

$F_g[\mathcal{S}_K(t_3,\dots,t_m,\dots)]$ is a generating function for $\left\langle \prod_i \psi_i^{d_i} \right\rangle$ — Kontsevich (92)

$\omega_k^{(g)}[\mathcal{S}_K(t_3,\dots,t_m,\dots)]$ is a generating function for $\left\langle \kappa_1 \dots \kappa_r \prod_{i=1}^{k} \psi_i^{d_i} \right\rangle$ — Eynard (07)

- For the generating function of $h_{g,\mu}$, totally expected :

  ◦ ELSV formula $$\frac{H_g(\mu)}{r(g,\mu)!} = \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}_{g,\ell(\mu)}}} \frac{\Lambda_g^{\vee}(1)}{\prod_{i=1}^{\ell(\mu)}(1-\mu_i\psi_i)}$$ Ekedahl, Lando, Shapiro, Vainhstein (01)

  $$\Lambda_g^{\vee}(w) = \sum_{j=0}^{g} \lambda_j\, w^j$$ total Chern class of the Hodge bundle of $\overline{\mathcal{M}_{g,\ell(\mu)}}$

  ◦ λ classes can be expressed in terms of ψ and κ classes — Mumford (83)

- Different approaches to simple Hurwitz numbers

simple covers of $\mathbf{P}^1$	cut-and-join formulas
'branched graphs'	generalization of Kontsevich model ?
group theoretical problem	matrix model

- In the topological vertex appears

triple Hodge integrals (trivalent vertices) $\longrightarrow$ proved to fit in the topological recursion for $\mathcal{S}_{\mathfrak{f}}\ :\ e^{\mathfrak{f}y} - e^{(\mathfrak{f}+1)y} - e^{x} = 0$

double Hurwitz numbers (propagators)

$\rightsquigarrow$ Towards matrix model techniques for GW of general toric CY

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Type A — Gromov-Witten invariants = intersection numbers in $\overline{\mathcal{M}_g}(\mathrm{X})$

$$\mathcal{M}_g(\mathrm{X}) = \{\text{holomorphic maps } \Sigma_g \xrightarrow{f} \mathrm{X}\}$$

$\overline{\mathcal{M}}_g(\mathrm{X}) =$ **compactification** of $\mathcal{M}_g(\mathrm{X})$ by allowing nodal points in Σ_g and **restriction to stable maps** (finite number of automorphisms)

- Exists for any symplectic CY 3-fold — virtual class construction : Kontsevich (96) ; Li, Tian (96) ; and many others

- When X is a toric CY 3-fold, $\int_{\overline{\mathcal{M}_g}(\mathrm{X})} \longrightarrow \int_{\overline{\mathcal{M}_{g,n}}}$ via localization techniques
  Graber, Pandharipande (99)

$$\mathcal{M}_{g,n} = \{(\Sigma_g, p_1, \ldots, p_n) \quad p_1, \ldots, p_n \in \Sigma_g\}$$

$\overline{\mathcal{M}_{g,n}} =$ **compatification** of $\mathcal{M}_{g,n}$ by allowing nodal points in Σ_g

$\Longrightarrow$ GW are encoded in Hodge integrals = intersection numbers in $\overline{\mathcal{M}_{g,n}}$